SYSTEMS OF HESS–APPEL'ROT TYPE AND ZHUKOVSKII PROPERTY

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Dedicated to the memory of Professor Novica Blažić (1959-2005)

Abstract. We start with a review of a class of systems with invariant relations, so called *systems of Hess–Appel'rot type* that generalizes the classical Hess–Appel'rot rigid body case. The systems of Hess–Appel'rot type have remarkable property: there exists a pair of compatible Poisson structures, such that a system is certain Hamiltonian perturbation of an integrable bi-Hamiltonian system. The invariant relations are Casimir functions of the second structure. The systems of Hess–Appel'rot type carry an interesting combination of both integrable and non-integrable properties.

Further, following integrable line, we study partial reductions and systems having what we call the *Zhukovskii property*: These are Hamiltonian systems on a symplectic manifold M with actions of two groups G and K; the systems are assumed to be K–invariant and to have invariant relation $\Phi = 0$ given by the momentum mapping of the G–action, admitting two type of reductions, a reduction to the Poisson manifold $P = M/K$ and a partial reduction to the symplectic manifold $N_0 = \Phi^{-1}(0)/G$; final and crucial assumption is that the partially reduced system to N_0 is completely integrable. We prove that the Zhukovskii property is a quite general characteristic of systems of Hess–Appel'rote type. The partial reduction neglects the most interesting and challenging part of the dynamics of the systems of Hess–Appel'rot type - the non-integrable part, some analysis of which may be seen as a reconstruction problem.

We show that an integrable system, the magnetic pendulum on the oriented Grassmannian $Gr^+(n, 2)$ has a natural interpretation within Zhukovskii property and that it is equivalent to a partial reduction of certain system of Hess–Appel'rot type. We perform a classical and algebro-geometric integration of the system in dimension four, as an example of a known isoholomorphic system - the Lagrange bitop.

The paper presents a lot of examples of systems of Hess–Appel'rot type, giving an additional argument in favor of further study of this class of systems.

2000 *Mathematics Subject Classification.* 70H06, 37J35, 53D25.

Contents

1. Introduction

Historically, the Hess-Appel'rot system as a classical rigid-body system, appeared (see [**35**]) just a year after the celebrated Kowalevski 1889 paper [**38**], and its immediate popularity had been connected with its relationship with the Kowalevski paper. Kowalevski started, as we know, from a careful analysis of the solutions of the Euler and the Lagrange case of rigid-body motion and formulated a problem *of describing the parameters* (A, B, C, x_0, y_0, z_0)*, for which the Euler–Poisson equations have a general solution in a form of uniform functions only with moving poles as singularities.* Here,

$I = \mathrm{diag}(A, B, C)$ represents the inertia operator, and $\chi = (x_0, y_0, z_0)$ is the centre of mass of the rigid body.

Then, in §1 of [**38**], after some necessary conditions were formulated, she descovered a new case, which is now known as the Kowalevski case. The last case was, according to Kowalevski, a unique possible beside the cases of Euler and Lagrange. However, considering the situation where all momenta of inertia are different, Kowalevski came to a relation analogue to the following (see [**34**]):

$$x_0\sqrt{A(B-C)} + y_0\sqrt{B(C-A)} + z_0\sqrt{C(A-B)} = 0,$$

and she concluded that $x_0 = y_0 = z_0$, which represented the Euler case.

However, Appel'rot noticed few years later, that the last relation admitted one more case, not observed in [**38**]:

$$x_0\sqrt{A(B-C)} + z_0\sqrt{C(A-B)} = 0, \quad y_0 = 0,$$

where he assumed $A > B > C$. Such intriguing position corresponding to the possible mistake in the Kowalevski paper, made the Hess-Appel'rot systems very attractive for leading Russian mathematicians from the end of XIX century as a possible counterexample. But, after a few years, Nekrasov and Lyapunov proved that the Hess-Appel'rot systems didn't satisfy the condition investigated by Kowalevski, which means that conclusion of §1 of [**38**] *was correct*.

And, from that moment until very recently, the Hess-Appel'rot systems were basically left aside, even in modern times, when new methods of inverse problems, Lax representations, finite-zone integrations were applied to almost all known classical systems.

A modern theory of systems of Hess-Appel'rot type has been developed in [**18, 19, 20, 21**]. It started with a construction of a Lax representation for the Hess-Appel'rot system in [**18**], see the Proposition 2.1 below. Generalization of this Lax pair in four-dimensional case led to construction of a new integrable rigid-body system in [**18**], called *the Lagrange bitop*. Algebro-geometric integration procedure of the Lagrange bitop has been performed in [**19**]. It brought to the discovery of a new class of integrable systems, which was named *isoholonomic systems* in [**19**]. Higher dimensional generalizations of the classical rigid-body Hess-Appel'rot systems have been constructed in [**20, 21**] as certain perturbations of the isoholonomic integrable systems. Finally, in [**21**] after detailed analysis of infinite set of new examples in arbitrary dimensions, the theory of systems of Hess-Appel'rot type has been settled down providing an axiomatic, general and abstract approach, see also the Section 3.

According to this theory, the systems of Hess-Appel'rot type form a class of dynamical systems, obtained as certain perturbations of integrable, bi-Hamiltonian systems which carry an interesting combination of both integrable and non-integrable properties.

Suppose a bi-Poisson structure $\{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_2$ is given, with an integrable, bihamiltonian system with the Hamiltonian H_0 corresponding to the first structure. Further, let $f_1, \dots, f_k$ be the commuting integrals of the system $(H_0, \{\cdot,\cdot\}_1)$, which are Casimirs for the second structure $\{\cdot,\cdot\}_2$. Then, a system of Hess-Appel'rot type is Hamiltonian with respect to the first structure with a Hamiltonian

$$H = H_0 + \sum_{l=1}^{k} J_l b_l f_l,$$

where J_l are constants and b_l are certain functions on the phase space. The invariant relations are

$$f_l = 0, \quad l = 1, \dots, k.$$

Thus, the invariant manifolds are symplectic leaves of the second Poisson structure.

As perturbations, they are global and not just small perturbations as it is usually the case in the study of non-integrable perturbations of integrable systems. A balance between integrable and nonintegrable properties is obtained by the choice of perturbations. The system of Hess-Appel'rot type is Hamiltonian with respect to the first Poisson structure, but perturbations and invariant relations are defined by Casimirs of the second structure.

For the classical Hess-Appel'rot case, its integrable part made it close to the Kowalevski study while its nonintegrable side finally disqualified it as a possible counterexample for the Kowalevski statement. Classical integration of its integrable part one may find also in the book of Golubev [**34**], while algebro-geometric integration has been performed in [**18**]. It was Zhukovskii who observed (see [**73**]), that after certain reduction, the classical Hess-Appel'rot system reduces to the completely integrable system of spherical pendulum (see the Subsection 5.1 below). This observation of Zhukovskii motivated us to introduce the notion of *Zhukovskii property*, see the Section see Section 5.

For the four-dimensional generalization of the Hess-Appel'rot system, detailed separation of integrable and nonintegrable part has been done in two ways in [**20, 21**], both classically and algebro-geometrically. Moreover, two integration procedures for the integrable part have been performed in all details in [**21**].

From these integrations, one can see that a completely integrable system in the same space requires one integration more in three-dimensional case and in the four-dimensional case it would require two integrations more.

Reviews of these results can be found in [**17**] and [**30**].

Following [**21**], the interest for the Zhukovskii property in the modern context has been expressed in [**36, 37**], where a study of partial reductions has been developed further, see Section 5.

One of the aims of the present paper is to provide a systematic study of the Zhukovski property from a general point of view of the axioms of the systems of Hess-Appel'rot type. It appears that the Zhukovskii property is a quite general characteristic of the systems of Hess-Appel'rot type and together with the partial reduction it traces well the integrable part of a system of Hess-Appel'rot type, see Sections 5, 6. In the same time, it neglects totally the most interesting and challenging part of the dynamics of the systems of Hess-Appel'rot type - the non-integrable part. This blindness to the non-integrable part, makes the partial reduction being important but of a limited range and domain in complete understanding and studying of the systems of Hess-Appel'rot type.

Nevertheless, a completely integrable system, the magnetic pendulum on $Gr^+(4,2)$ which has been introduced and studied in [**14**] within the study of magnetic flows on homogeneous spaces, appears to have an interpretation within Zhukovski property: we show that it can be obtained as a partial reduction of a certain system of Hess-Appel'rot type. Moreover, it appears that the magnetic pendulum is equivalent to a very simple instant of the Lagrange bitop from [**19**]. Thus, the magnetic pendulum is an example of an isoholonomic system. So, the integration techniques of [**19**] and [**21**] can bee applied directly to the magnetic pendulum. Complete integration of the magnetic pendulum in classical and algebro-geometric manner is the second main aim of this paper and it is performed in the Section 8.

The paper provides a lot of examples of systems of Hess-Appel'rot type, motivated from [**20, 21, 36, 37, 23**] and from references therein, see the Sections 4 and 7. Such a rich set of examples is additional argument in favor of further study of the systems of Hess-Appel'rot type.

At the end of the paper, we collected for a reader's sake all necessary notions from the theory of Hamiltonian systems and their reductions, see the Appendix.

2. Classical three-dimensional Hess-Appel'rot system

The Euler-Poisson equations of the motion of a heavy rigid body in the moving frame are [**34**]:

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times \mathbf{\Omega} + \mathbf{\Gamma} \times \chi, \\ \dot{\mathbf{\Gamma}} &= \mathbf{\Gamma} \times \mathbf{\Omega} \\ \mathbf{\Omega} &= \tilde{J}\mathbf{M}, \quad \tilde{J} = \operatorname{diag}(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3), \end{aligned} \tag{2.1}$$

where $\mathbf{M}$ is the kinetic momentum vector, $\mathbf{\Omega}$ the angular velocity, $\tilde{J}$ a diagonal matrix, the inverse of inertia operator, $\mathbf{\Gamma}$ a unit vector fixed in the space and χ is the radius vector of the centre of masses.

It is well known (see for example [**34**]) that equations (2.1) have three integrals of motion:

$$F_1 = \frac{1}{2}\langle \mathbf{M}, \mathbf{\Omega}\rangle + \langle \mathbf{\Gamma}, \chi\rangle, \qquad F_2 = \langle \mathbf{M}, \mathbf{\Gamma}\rangle, \qquad F_3 = \langle \mathbf{\Gamma}, \mathbf{\Gamma}\rangle = 1. \tag{2.2}$$

Thus, for complete integrability, one integral more is necessary [**34**]. Let $\tilde{J}_1 < \tilde{J}_2 < \tilde{J}_3$ and $\chi = (x_0, y_0, z_0)$. Hess in [**35**] and Appel'rot in [**4**] found that if the inertia momenta and the radius vector of the centre of masses satisfy the conditions

$$\begin{gathered} y_0 = 0 \\ x_0\sqrt{\tilde{J}_3 - \tilde{J}_2} + z_0\sqrt{\tilde{J}_2 - \tilde{J}_1} = 0, \end{gathered} \tag{2.3}$$

then the surface

$$F_4 = M_1 x_0 + M_3 z_0 = 0 \tag{2.4}$$

is invariant.

The compact connected components of the regular invariant sets given by (2.2), (2.4) are tori, but not with quasi-periodic dynamics. The classical and algebro-geometric integration can be found in [**34**] and [**18**], respectively. It is shown that the equations of the motion reduce to one elliptic integral and one Riccati differential equation.

The Zhukovskii geometric interpretation of the conditions (2.3) [**73, 40**]. Let us consider the ellipsoid

$$\frac{M_1^2}{\tilde{J}_1} + \frac{M_2^2}{\tilde{J}_2} + \frac{M_3^2}{\tilde{J}_3} = 1,$$

and the plane containing the middle axis and intersecting the ellipsoid at a circle. Denote by l the normal to the plane, which passes through the fixed point O. Then the condition (2.3) means that the centre of masses lies on the line l.

Having this interpretation in mind, we choose a basis of moving frame such that the third axis is l, the second one is directed along the middle axis of the ellipsoid, and the first one is chosen according to the orientation of the orthogonal frame. In this basis (see [**15**]), the particular integral (2.4) becomes

$$F_4 = M_3 = 0,$$

and the matrix $\tilde{J}$ and mass centre χ obtain the form:

$$J = \begin{pmatrix} J_1 & 0 & J_{13} \\ 0 & J_1 & 0 \\ J_{13} & 0 & J_3 \end{pmatrix}, \qquad \chi = (0, 0, z_0). \tag{2.5}$$

This will serve us as a motivation for a definition of the four-dimensional Hess-Appel'rot system.

A three-dimensional Lagrange top is defined by the Hamiltonian:

$$H_L = \frac{1}{2}\left(\frac{M_1^2 + M_2^2}{I_1} + \frac{M_3^2}{I_3}\right) + z_0\Gamma_3,$$

according to the standard Poisson structure

$$\{M_i, M_j\}_1 = -\epsilon_{ijk}M_k,\ \{M_i, \Gamma_j\}_1 = -\epsilon_{ijk}\Gamma_k,\ \{\Gamma_i, \Gamma_j\} = 0$$

on the Lie algebra $e(3)$. It is also well-known that three-dimensional Lagrange top is Hamiltonian in another Poisson structure, compatible with first one. This structure is defined by:

$$\{\Gamma_i, \Gamma_j\}_2 = -\epsilon_{ijk}\Gamma_k,\ \{M_1, M_2\}_2 = 1,$$

and the corresponding Hamiltonian is:

$$\tilde{H}_L = (a-1)M_3\left(\frac{1}{2}(M_1^2 + M_2^2) + \Gamma_3\right) + M_1\Gamma_1 + M_2\Gamma_2 + M_3\Gamma_3$$

where $I_1 = 1, I_3 = a, z_0 = 1$. Casimir functions in the second structure are $\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2$ and M_3.

Let us observe that the Hamiltonian for the three-dimensional Hess-Appel'rot case is a quadratic deformation of Hamiltonian H_L of the Lagrange top:

$$H_{HA} = H_L + J_{13}M_1M_3.$$

The function M_3, which gives the invariant relation for the Hess-Appel'rot case, is a Casimir function of the second Poisson structure.

Let us mention that the Lax representation for Hess-Appel'rot system (2.1), (2.3), (2.4) is constructed in [**18**]:

Proposition 2.1. ([**18**]) *On invariant manifold given by the invariant relation* (2.4)*, the equations of Hess-Appel'rot system are equivalent to the matrix equation*

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)]$$

where $L(\lambda) = \lambda^2 C + \lambda M + \Gamma,\ A(\lambda) = \lambda\chi + \Omega.$

Here we denoted with $M \in so(3)$ antisymmetric matrix that corresponds to the vector $\mathbf{M} \in \mathbb{R}^3$ due to correspondence $M_{ij} = -\epsilon_{ijk}M_k$ (and similar for C, Γ, Ω, χ), and $C = \frac{1}{J_2}\chi$. Another Lax representation for the Hess-Appel'rot system is given in [**22**]. Using it a sort of separation of variables for Hess-Appel'rot system is discussed there. By putting $n = 3$ in (4.12) we get yet another Lax representation (see [**36**]).

3. General systems of Hess-Appel'rot type. Axiomatic approach

3.1. The first set of axioms: general Poisson settings. Suppose a Poisson manifold $(M^{2n}, \{\cdot,\cdot\})$ is given, together with $k+1$ functions $H, f_1, \ldots, f_k \in C^\infty(M)$, such that

(A1)

$$\{H, f_i\} = \sum_{j=1}^{k} a_{ij} f_j, \quad a_{ij} \in C^\infty(M), \quad i, j = 1, \dots, k;$$

(A2)

$$\{f_i, f_j\} = 0, \quad i, j = 1, \dots, k.$$

A more general case can be obtained by replacing condition (A2) with

(A2')

$$\{f_i, f_j\} = \sum_{l=1}^{k} d_{ij}^l f_l, \quad d_{ij}^l = const, \quad i, j = 1, ..., k.$$

In this case, the algebra of invariant relations is a noncommutative Lie algebra.

Starting from the Hamiltonian system (M, H_0) with k integrals in involution $f_1, \dots, f_k$, choosing functions $b_j \in C^\infty(M), \quad j = 1, \dots, k$, one comes to a restrictively integrable system:

(HP) **(Hamiltonian perturbation axiom)**

The system (M, H) where

$$H = H_0 + \sum_{j=1}^{k} b_j f_j,$$

will be called a Hamiltonian perturbation. It satisfies (A1) with

$$a_{ij} = \{b_j, f_i\}, \quad i, j = 1, \dots, k.$$

(BP) **(Bi-Poisson axiom)** *There exist a pair of compatible Poisson structures, such that the system is Hamiltonian with respect to the first structure, having the Hamiltonian of the form* (HP)*, such that f_i are Casimir functions with respect to the second structure.*

The invariant relations define symplectic leaves with respect to the second structure, and the system is Hamiltonian with respect the first one.

3.2. The second set of axioms: Kowalevski property. To get the right choice of axioms, we have to turn back to the Kowalevski analysis. First, we are going to introduce some general notions, see [**39**].

Suppose a system of ODEs of the form

$$\dot{z}_i = f_i(z_1, \dots, z_n), \quad i = 1, \dots, n, \tag{3.1}$$

is given and there exist positive integers $g_i, \quad i = 1, \dots, n$, such that

$$f_i(a^{g_1} z_1, \dots, a^{g_n} z_n) = a^{g_i+1} f_i(z_1, \dots, z_n), \quad i = 1, \dots, n.$$

Then the system (3.1) is *quasi-homogeneous* and numbers g_i are *exponents of quasi-homogeneity.* Then, for any complex solution $C = (C_1, \dots, C_n)$ of the system of algebraic equations:

$$-g_i C_i = f_i(C_1, \dots, C_n), \quad i = 1, \dots, n, \tag{3.2}$$

one can define *the Kowalevski matrix* $K = K(C) = [K_j^i(C)]$:

$$K_j^i(C) = \frac{\partial f_i}{\partial z_j}(C) + g_i \delta_j^i.$$

Eigen-values of the Kowalevski matrix are called *the Kowalevski exponents.* This terminology was introduced in [**71**]. In last twenty years, heuristic and theoretical methods in application of Kowalevski matrix and Kowalevski exponents in study of integrability and nonintegrability have been actively developing, see for example [**2, 3, 39**]. But, the notion of Kowalevski matrix and Kowalevski exponents were introduced by Kowalevski herself in [**38**]. The criterion she used (see [**38**], p. 183, l. 15-22) to detect a system which is now known as the Kowalevski top, can be formulated in Yoshida terminology as:

Kowalevski condition (Kc). *The* 6×6 *Kowalevski matrix should have* five different positive integer *Kowalevski exponents.*

Now we return to the study of systems of Hess-Appel'rot type. The systems we have constructed *are quasi-homogeneous.* Exponents of each M variable are $g=1$, and for any Γ they are equal to two. We are going now to calculate Kowalevski exponents for the Hess-Appel'rot systems.

Three-dimensional Hess-Appel'rot case. Let us denote $(M_1, M_2, M_3, \Gamma_1, \Gamma_2, \Gamma_3)$ by $(z_1, \dots, z_6)$. Then the Euler-Poisson equations take the form (3.1) with

$$\begin{aligned}
f_1 &= (J_3 - J_1) z_2 z_3 + J_{13} z_1 z_2 + z_5;\\
f_2 &= (J_3 - J_1) z_1 z_3 + J_{13}(z_3^2 - z_1^2) - z_4;\\
f_3 &= J_{13} z_2 z_3;\\
f_4 &= J_3 z_5 z_3 - J_1 z_2 z_6 + J_{13} z_1 z_5;\\
f_5 &= -J_3 z_3 z_4 + J_1 z_1 z_6 + J_{13}(z_3 z_6 - z_1 z_4);\\
f_6 &= J_3 z_2 z_4 - J_1 z_1 z_5 - J_{13} z_3 z_5;
\end{aligned}$$

and $g_i = 1, \quad i = 1,2,3$ and $g_i = 2, \quad i = 4,5,6$. The invariant relation corresponds to the constraint $c_3 = 0$. So, we are looking for solutions $(c_1, c_2, 0, c_4, c_5, c_6)$ of the system of the form (3.2). One can easily get $c_4 = -J_{13} c_1^2 + c_2$, $c_5 = -c_1(1 + J_{13} c_2)$, $c_6 = -(c_1^2 + c_2^2)/2$. Then, for $c_1 \neq 0$, we get four possible solutions for (c_1, c_2) divided into two pairs: $(\pm i/J_{13}, -1/J_{13})$ and $(\pm 2i/J_{13}, -2/J_{13})$. The Kowalevski exponents are

$$(-1,-2,2,4,3,3), \qquad (-1,1,3,2,2,2),$$

respectively.

Thus, it can easily be seen that classical Hess-Appel'rot system doesn't satisfy exactly the Kowalevski condition (Kc), although it is quite close to.

Thus, using into account properties of Kowalevski exponents of algebraically-integrable Hamiltonian systems, we can conclude that for the systems we have constructed, functions b_i in the perturbation formula (HP) should satisfy two conditions:

(QH) **(quasi-homogeneity axiom)** *The obtained system of Hamiltonian equations has to be quasi-homogeneous.*

In such a case, a Kowalevski matrix exists and we come to the last condition. Suppose the invariant relations correspond to equations $z_1 = 0, \dots, z_k = 0$.

Denote by p number of Casimirs: $n = p + 2m$, where $2m$ is the dimension of a general symplectic leaf.

(ArA) **(Arithmetic axiom)** *For any nonzero solution* $C = (0, ..., 0, c_{k+1}, ..., c_n)$ *of the system* (3.2)*, the Kowalevski matrix* $K(C)$ *has* $n-p$ *eigen-vectors tangent to the symplectic leaf and* p *transversal to it. Half of the Kowalevski exponents which correspond to tangential eigen-vectors and all of transversal ones are rational numbers. Irrational numbers among the second half of tangential Kowalevski*

exponents are divided into pairs such that the differences are integrally dependent.

The axioms of systems of Hess-Appel'rot type provide conditions which determine classical Hess-Appel'rot system among three-dimensional systems of Hess-Appel'rot type. More precisely, suppose the two Poisson brackets are given on $e(3)$ as above and a system is given by a Hamiltonian

$$H_1 = H_0 + JbM_3, \tag{3.3}$$

where H_0 is the Hamiltonian of the Lagrange top corresponding to the first Poisson structure, M_3 is its integral and a Casimir for the second structure, J is a nonzero constant and b is a function, such that the axioms of the systems of Hess-Appel'rot are satisfied.

In [**21**] the following *rigidity theorem* has been proved.

Theorem 3.1 ([**21**])**.** *The only non-zero polynomials b which give systems of Hess-Appel'rot type by relation* (3.3) *are of the form*

$$b(z_1, \dots, z_6) = z_1 + kz_3.$$

All systems of Hess-Appel'rot type of the form (3.3) *are the classical Hess-Appel'rot systems.*

The last theorem provides a strong argument in favor of the choice of axioms which have been postulated in [**21**]. We take these axioms as the starting point of our current research.

Since the Zhukovskii property is main object of our study in this paper, we are going to focus ourselves on the first set of axioms.

4. Examples of systems of Hess-Appel'rot type

4.1. Rigid body systems on $so(n)\times so(n)$. The Euler-Poisson equations of motion of a heavy rigid body fixed at a point are Hamiltonian on the Lie algebra $e(3)$, which is the semi-direct product of Lie algebras $\mathbb{R}^3$ and $so(3)$. Since $\mathbb{R}^3$ is isomorphic to $so(3)$, there are two natural higher-dimensional generalizations of Euler-Poisson equations. The first one is given by Ratiu in [**64**] and it is to the semi-direct product $so(n) \times so(n)$ and the second one is to the Lie algebra $e(n) = \mathbb{R}^n \times so(n)$.

Equations of a heavy n-dimensional rigid body on $so(n) \times so(n)$ are :

$$\begin{aligned} \dot{M} &= [M, \Omega] + [\Gamma, \chi] \\ \dot{\Gamma} &= [\Gamma, \Omega], \end{aligned} \tag{4.1}$$

where $M, \Omega, \Gamma, \chi \in so(n)$, $\Omega = AM$ and χ is a constant matrix (see [**64**]). Here $A : so(n) \to so(n)$ is the inverse of the rigid body kinetic energy operator. The Euler-Poisson equations (4.1) are Hamiltonian with the Hamiltonian function

$$H = \frac{1}{2}\langle M, \Omega\rangle + \langle \chi, \Gamma\rangle = -\frac{1}{4}\mathrm{tr}(M\Omega) - \frac{1}{2}\mathrm{tr}(\chi\Gamma), \tag{4.2}$$

in the standard Poisson structure on the semi-direct product $so(n) \times so(n)$:

$$\{M_{ij}, M_{jk}\}_1 = -M_{ik}, \qquad \{M_{ij}, \Gamma_{jk}\}_1 = -\Gamma_{ik}, \qquad \{\Gamma_{ij}, \Gamma_{kl}\}_1 = 0.$$

The Casimir functions are $\mathrm{tr}(\Gamma^{2k})$, $\mathrm{tr}(M\Gamma^{2k+1})$ and the dimension of generic symplectic leaf is $n(n-1) - 2\left[\frac{n}{2}\right]$.

We will suppose that

$$\Omega = JM + MJ, \tag{4.3}$$

where J is a constant symmetric matrix. The operator $M \mapsto JM + MJ$ belongs to the class of Manakov operators [**50**] on $so(n)$.

The Lax representation together with Zhukovskii geometric interpretation presented in the second section, where inspiration for a construction of a higher-dimensional generalization of Hess-Appel'rot system in [**21**]. Let us first consider the four-dimensional case.

Definition 4.1. The four-dimensional Hess-Appel'rot system is described by the equations (4.1), (4.3) and conditions:

$$J = \begin{pmatrix} J_1 & 0 & J_{13} & 0 \\ 0 & J_1 & 0 & J_{24} \\ J_{13} & 0 & J_3 & 0 \\ 0 & J_{24} & 0 & J_3 \end{pmatrix}, \qquad \chi = \begin{pmatrix} 0 & \chi_{12} & 0 & 0 \\ -\chi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi_{34} \\ 0 & 0 & -\chi_{34} & 0 \end{pmatrix}, \tag{4.4}$$

such that the operator $M \mapsto JM + MJ$ is positive definite and $\chi_{12}^2 + \chi_{34}^2 \neq 0$.

The invariant surfaces are determined in the next lemma.

Lemma 4.1. (i) *For the four-dimensional Hess-Appel'rot system, the following relations take place:*

$$\dot{M}_{12} = J_{13}(M_{13}M_{12} + M_{24}M_{34}) + J_{24}(M_{13}M_{34} + M_{12}M_{24}),$$
$$\dot{M}_{34} = J_{13}(-M_{13}M_{34} - M_{12}M_{24}) + J_{24}(-M_{13}M_{12} - M_{24}M_{34}).$$

(ii) *The system has two invariant relations:*

$$M_{12} = 0, \qquad M_{34} = 0. \tag{4.5}$$

Now let us introduce a new Poisson structure, compatible with the standard one, as follows:

$$\begin{array}{lll} \{\Gamma_{ij}, \Gamma_{jk}\}_2 = -\Gamma_{ik}, & \{M_{ij}, \Gamma_{kl}\}_2 = 0, & \{M_{13}, M_{23}\}_2 = -\chi_{12}, \\ \{M_{14}, M_{24}\}_2 = -\chi_{12}, & \{M_{13}, M_{14}\}_2 = -\chi_{34}, & \{M_{23}, M_{24}\}_2 = -\chi_{34}. \end{array} \tag{4.6}$$

Casimir functions in this structure are M_{12}, M_{34}, $\Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{14}^2 + \Gamma_{23}^2 + \Gamma_{24}^2 + \Gamma_{34}^2$, and $\Gamma_{12}\Gamma_{34} + \Gamma_{23}\Gamma_{14} - \Gamma_{13}\Gamma_{24}$.

The situation with four-dimensional Hess-Appel'rot case is similar to the three-dimensional case: the Hamiltonian is again a quadratic deformation:

$$H_{HA} = H_{LB} + J_{13}(-M_{12}M_{23} + M_{14}M_{34}) + J_{24}(M_{12}M_{14} - M_{23}M_{34}),$$

where $H_{LB} = \frac{1}{2}(2J_1M_{12}^2 + (J_1+J_3)M_{13}^2 + (J_1+J_3)M_{14}^2 + (J_1+J_3)M_{23}^2 + (J_1+J_3)M_{24}^2 + 2J_3M_{34}^2) + \chi_{12}\Gamma_{12} + \chi_{34}\Gamma_{34}$ is the Hamiltonian function of the *Lagrange bitop*. The Lagrange bitop is a complete integrable system of a heavy rigid body on $so(4) \times so(4)$ defined in [**18**] and studied in details in [**19**]. Moreover, the Lagrange bitop is a bi-Hamiltonian system. Assume that $J_1 = a$, $J_3 = 1 - a$. Then

$$\begin{aligned} \tilde{H}_{LB} =& \frac{(2a-1)(\chi_{12}M_{12} + \chi_{34}M_{34})}{\chi_{12}^2 - \chi_{34}^2}\left(\frac{M_{13}^2 + M_{14}^2 + M_{23}^2 + M_{24}^2}{2} + \chi_{12}\Gamma_{12} + \chi_{34}\Gamma_{34}\right) \\ &+ \frac{(1-2a)(\chi_{12}M_{34} + \chi_{34}M_{12})}{\chi_{12}^2 - \chi_{34}^2}(M_{23}M_{14} - M_{13}M_{24} + \chi_{12}\Gamma_{34} + \chi_{34}\Gamma_{12}) \\ &+ M_{12}\Gamma_{12} + M_{13}\Gamma_{13} + M_{14}\Gamma_{14} + M_{23}\Gamma_{23} + M_{24}\Gamma_{24} + M_{34}\Gamma_{34}. \end{aligned}$$

is the Hamiltonian in the second structure (4.6). The functions M_{12} and M_{34}, giving invariant relations for the four-dimensional Hess-Appel'rot system, are Casimir functions for the Poisson structure (4.6).

Let us pass to the arbitrary dimension $n > 4$.

Definition 4.2. The n-dimensional Hess-Appel'rot system is described by the equations (4.1), (4.3), together with conditions:

(4.7)
$$
\begin{aligned}
&J = \mathrm{diag}(J_1, J_1, J_3, \dots, J_3) + J_{13}(E_1 \otimes E_3 + E_3 \otimes E_1) + J_{24}(E_2 \otimes E_4 + E_4 \otimes E_2),\\
&\chi = \chi_{12} E_1 \wedge E_2, \qquad \chi_{12} \neq 0.
\end{aligned}
$$

Invariant relations are given in next lemma.

Lemma 4.2. (i) *For the n-dimensional Hess-Appel'rot system we have:*

$$
\begin{aligned}
\dot{M}_{12} =& J_{13}(M_{12}M_{13} + M_{24}M_{34} + \sum_{p=5}^{n} M_{2p}M_{3p}) + \\
& J_{24}(M_{12}M_{24} + M_{13}M_{34} - \sum_{p=5}^{n} M_{1p}M_{4p}) \\
\dot{M}_{34} =& -J_{13}(M_{13}M_{34} + M_{24}M_{12} + \sum_{p=5}^{n} M_{1p}M_{p4}) - \\
& J_{24}(M_{13}M_{12} + M_{24}M_{34} + \sum_{p=5}^{n} M_{2p}M_{3p}) \\
\dot{M}_{3p} =& -J_{13}(M_{13}M_{3p} + M_{2p}M_{12}) - J_{24}(M_{34}M_{2p} + M_{23}M_{4p}) + \\
& M_{34}\Omega_{4p} - \Omega_{34}M_{4p} + \sum_{k=5}^{n}(M_{3k}\Omega_{kp} - \Omega_{3k}M_{4p}), \qquad p > 4, \\
\dot{M}_{4p} =& J_{13}(-M_{14}M_{3p} + M_{1p}M_{34}) + J_{24}(M_{12}M_{1p} - M_{24}M_{4p}) - \\
& M_{34}\Omega_{3p} + \Omega_{34}M_{3p} + \sum_{k=5}^{n}(M_{4k}\Omega_{kp} - \Omega_{4k}M_{4p}), \qquad p > 4, \\
\dot{M}_{kl} =& 0, \qquad k, l > 4
\end{aligned}
$$

(ii) *The system has the following set of invariant relations:*

$$
M_{12} = 0, \qquad M_{lp} = 0, \qquad l, p \geq 3. \tag{4.8}
$$

In [**64**] the n-dimensional *Lagrange top* on the semidirect product $so(n) \times so(n)$ is constructed. In the metric $\Omega = JM + MJ$, where $J = \mathrm{diag}(J_1, J_1, J_3, \dots, J_3)$, the n-dimensional Lagrange top is defined with Hamiltonian

$$
H_L = \frac{1}{2}\left(2J_1 M_{12}^2 + (J_1 + J_3)\sum_{p=3}^{n}(M_{1p}^2 + M_{2p}^2) + 2J_3 \sum_{3 \leqslant p < q \leqslant n} M_{pq}^2\right) + \chi_{12}\Gamma_{12}
$$

Consider a Poisson structure

$$
\{\Gamma_{ij}, \Gamma_{jk}\}_2 = -\Gamma_{ik},\ \{M_{ij}, M_{kl}\}_2 = 0,\ \{M_{1l}, M_{2l}\}_2 = -1, \quad l = 3, \dots, n, \tag{4.9}
$$

compatible with the standard one. The dimension of a symplectic leaf in this structure is $\frac{(n-2)(n-3)}{2} - \left[\frac{n-2}{2}\right] + 4(n-2)$, hence there are $\frac{n^2-5n+8}{2} + \left[\frac{n}{2}\right]$ Casimir functions:

$$
M_{12}, \quad M_{pq}, \quad \mathrm{tr}(\Gamma^{2k}), \qquad 2 < p < q \leq n, \quad k = 1, \dots, \left[\frac{n}{2}\right].
$$

The n-dimensional Lagrange top is also bi-Hamiltonian system. In the Poisson structure (4.9) its Hamiltonian is:

$$\tilde{H}_L = (2a-1)M_{12}\left(\frac{1}{2}\sum_{p=3}^{n}(M_{1p}^2+M_{2p}^2)+\Gamma_{12}\right)+$$
$$(1-2a)\sum_{3\leqslant p<q\leqslant n} M_{pq}(M_{1q}M_{2p}-M_{2q}M_{1p}+\Gamma_{pq})+\sum_{1\leqslant p<q\leqslant n} M_{pq}\Gamma_{pq},$$

where $J_1 = a,\ J_3 = 1-a,\ \chi_{12} = 1$.

Similarly as in dimension 3 and 4, Hamiltonian for the Hess-Appel'rot system in arbitrary dimension n is a quadratic deformation of the Hamiltonian for the n-dimensional Lagrange top:

$$H_{HA} = H_L + \sum_{k=1}^{n}(J_{13}M_{1k}M_{3k}+J_{24}M_{2k}M_{4k}),$$

and functions $M_{12}, M_{pq}, p, q \geqslant 3$, which give the invariant relations, are Casimir functions for the Poisson structure (4.9)

Next theorem gives a Lax pair for the Hess-Appel'rot system.

Theorem 4.1 ([**21**])**.** *On invariant manifold given by the invariant relations, the equations of n-dimensional Hess-Appel'rot system are equivalent to the matrix equation*

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)], \qquad L(\lambda) = \lambda^2\frac{1}{J_1+J_3}\chi + \lambda M + \Gamma, \qquad A(\lambda) = \lambda\chi + \Omega.$$

Using this Lax representation, the both classical and algebro-geometric integration procedures are presented in [**21**] in dimension four.

4.2. Rigid body systems on $e(n)$. Let us now consider rigid body motion on the Lie algebra $e(n)$. The standard Poisson structure on $e(n)^* \cong e(n)(M,\Gamma)$ is given with:

$$\{M_{ij}, M_{kl}\}_1 = -\delta_{jk}M_{il}, \qquad \{\Gamma_i, \Gamma_j\}_1 = 0, \qquad \{M_{ij}, \Gamma_k\}_1 = -\Gamma_i\delta_{jk} + \Gamma_j\delta_{ik}.$$

Here we identified $e(n)^* \cong e(n)$ by the use of a non-invariant scalar product:

$$\langle (M,\Gamma),(M,\Gamma)\rangle = \langle M, M\rangle + \langle \Gamma,\Gamma\rangle = -\frac{1}{2}\mathrm{tr}(MM) + \sum_{i=1}^{n}\Gamma_i^2.$$

A heavy rigid body Hamiltonian read: $H = \frac{1}{2}\langle M,\Omega\rangle + \langle \chi, \Gamma\rangle$, where $\chi \in \mathbb{R}^n$ is the vector of mass centre and $\Gamma \in \mathbb{R}^n$ is a vertical vector considered in the moving coordinate system and, as above, $\Omega = AM$. We can choose $\chi = \chi_n E_n = (0,\dots,0,\chi_n)$. The corresponding Euler–Poisson equations are:

$$\begin{aligned} \dot{M} &= [M,\Omega] + \chi_n E_n \wedge \Gamma \\ \dot{\Gamma} &= -\Omega\cdot\Gamma. \end{aligned} \tag{4.10}$$

In [**7**] Belyaev considered the n-dimensional *Lagrange top* defined by the Hamiltonian function:

$$H_\Lambda = \frac{a}{2}\langle M_{\mathfrak{d}}, M_{\mathfrak{d}}\rangle + \frac{b}{2}\langle M_{\mathfrak{g}}, M_{\mathfrak{g}}\rangle + \chi_n\Gamma_n, \tag{4.11}$$

where $so(n) = \mathfrak{g}\oplus\mathfrak{d}$ is orthogonal symmetric-pair decomposition of $so(n)$: $\mathfrak{g} = \langle E_i \wedge E_j \,|\, 1 \le i < j \le n-1\rangle \cong so(n-1)$, $\mathfrak{d} = \langle E_i\wedge E_n \,|\, 1\le i\le n-1\rangle$ and $a, b > 0$ are real parameters. Note that the kinetic energy has the Manakov form (4.3), where we take $\Omega = J_\Lambda M + MJ_\Lambda$, $J_\Lambda = \mathrm{diag}(J_1, J_1, \dots, J_1, J_n)$. Then $a = J_1 + J_n$, $b = 2J_1$.

Belyaev proved noncommutative integrability of the system [**7**]. The Lax representation is given by Reyman and Semenov-Tian-Shanski [**66**]:

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)], \quad L(\lambda) = \hat{\Gamma} + \lambda \hat{M} + \lambda^2 \frac{\chi_n}{a} \hat{E}_n, \quad A(\lambda) = \hat{\omega} + \lambda \chi \hat{E}_n. \tag{4.12}$$

Here, for a given $(M, \Gamma) \in e(n)$, $\hat{M}, \hat{\Gamma} \in so(n+1)$ are defined by

$$\hat{M} = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\Gamma} = \begin{pmatrix} \mathbf{0} & \Gamma \\ -\Gamma^t & 0 \end{pmatrix}.$$

Let $\mathcal{L}$ be the set of integrals obtained from (4.12)

$$\mathcal{L}: \qquad \operatorname{tr}(L(\lambda)^{2k}), \qquad k = 1, \dots, \operatorname{rank} SO(n+1), \qquad \lambda \in \mathbb{R} \tag{4.13}$$

and let $\mathcal{S}$ be linear functions on $\mathfrak{g} = so(n-1)$

$$\mathcal{S} = \{M_{ij} \mid 1 \le i < j \le n-1\}. \tag{4.14}$$

Then $\{\mathcal{L}, \mathcal{L}\}_1 = 0$, $\{\mathcal{L}, \mathcal{S}\}_1 = 0$ and $\mathcal{L} + \mathcal{S}$ is complete set of integrals of the Lagrange top system (4.10), (4.11).

As above, let us define a Hess-Appel'rot system on $e(n)$ as a perturbation of the Lagrange top using operator (4.3) with

$$J = J_\Lambda + J_\Pi, \quad J_\Lambda = \operatorname{diag}(J_1, J_1, \dots, J_1, J_n), \quad J_\Pi = J_{1n}(E_1 \otimes E_n + E_n \otimes E_1). \tag{4.15}$$

Proposition 4.1. *The equations* (4.10), (4.3), (4.15) *have invariant relations:*

$$M_{ij} = 0, \qquad 1 \le i < j \le n-1. \tag{4.16}$$

Restriction of the system on invariant manifold is given with:

$$\begin{aligned}
&\dot{M}_{in} = -\sum_{j=1}^{n-1} \Omega^{\Pi}_{ij} M_{jn} - \chi_n \Gamma_i, \quad i = 1, \dots, n-1, \\
&\dot{\Gamma}_i = -(J_1 + J_n)\Gamma_n M_{in} - \sum_{j=1}^{n-1} \Omega^{\Pi}_{ij} \Gamma_j, \quad i = 1, \dots, n-1, \\
&\dot{\Gamma}_n = (J_1 + J_n) \sum_{j=1}^{n-1} \Gamma_j M_{jn},
\end{aligned} \tag{4.17}$$

where $\Omega_\Pi = M J_\Pi + J_\Pi M = -J_{1n} \sum_{i=2}^{n-1} (M_{in} E_1 \wedge E_i + M_{1i} E_i \wedge E_n)$.

Remark 4.1. It can be proved that equations (4.10) have an invariant relation (4.16) if and only if $\mathrm{pr}_{\mathfrak{d}} \circ A \circ \mathrm{pr}_{\mathfrak{d}}$ is proportional to the identity operator on $\mathfrak{d}$ [**36**].

Applying a general construction of compatible Poisson structures related to the symmetric pair decomposition of semi-simple Lie algebras for the symmetric pair $(so(n+1), so(n))$ (see [**66, 70, 10**]), one gets the second Poisson structure on $e(n)$:

$$\{\Gamma_i, M_{jn}\}_2 = \delta_{ij}, \qquad \{\Gamma_i, \Gamma_j\}_2 = -M_{ij}. \tag{4.18}$$

Let us denote $\chi_n = a = J_1 + J_n = 1$, $b = 2J_1$.

Proposition 4.2. (i) *The Casimirs of the structure* (4.18) *are functions* (4.14) *together with* $I = \frac{1}{2}(M_{1n}^2 + M_{2n}^2 + \cdots + M_{n-1,n}^2) + \Gamma_n$.

(ii) *The Lagrange top is Hamiltonian with respect to the Poisson structure* (4.18) *and the Hamiltonian function:*

$$H_{\Lambda,2} = \frac{1}{2}(\Gamma, \Gamma) + (b-1) \sum_{i,j=1}^{n-1} \Gamma_i M_{ij} M_{jn}.$$

In this setting, the system is integrable in the commutative sense by means of integrals (4.13)*. The invariant tori are of dimension* $n-2$*.*

Therefore, the system (4.10), (4.3), (4.15) satisfies axioms (A1), (A2'), (HP) and (BP).

The restricted Hess-Appel'rot system (4.17) admits the Lax representation (4.12) with M and Ω related by (4.3), (4.15). But, only 3 integrals from the family (4.13) are independent when invariant relations (4.16) are satisfied. Namely, the spectral curve is then given by [**36**]:

$$p(\lambda,\mu) = \det(L(\lambda)-\mu\mathrm{Id}) = (-\mu)^{n-3}\left(\mu^4+\mu^2 P(\lambda)+Q(\lambda)\right) = 0, \tag{4.19}$$
$$P(\lambda) = F_2 + \frac{2}{a}\lambda^2 F_1 + \lambda^4\left(\frac{\chi_n}{a}\right)^2, \quad Q(\lambda) = \lambda^2 F_3,$$

where F_1, F_2, F_3 are integrals

$$F_1 = H = \frac{a}{2}\sum_{i=1}^{n-1} M_{in}^2 + \chi_n\,\Gamma_n, \quad F_2 = \sum_{i=1}^{n}\Gamma_i^2 = 1,$$
$$F_3 = \sum_{1\le i<j\le n-1}(M_{in}\Gamma_j - M_{jn}\Gamma_i)^2, \tag{4.20}$$

that correspond to the integrals (2.2) of the classical problem.

On the invariant submanifold (4.16), integrals F_1, F_2, F_3 are not enough to give an integrability of the Lagrange top. But, when conditions (4.16) are satisfied, the system (4.10), (4.11) has additional integrals

$$F_{ij} = M_{in}\Gamma_j - M_{jn}\Gamma_i, \qquad 1\le i<j\le n-1$$

that implies that the Lagrange top on invariant submanifold (4.16) can be solved by quadratures: (4.16) is almost everywhere foliated on invariant 2-dimensional tori.

Integrals F_{ij} are not integrals of Hess-Appel'rot system (4.17). Nevertheless, the system (4.17) has additional invariant relations

$$F_{ij} = M_{in}\Gamma_j - M_{jn}\Gamma_i = 0, \qquad 1\le i<j\le n-1, \tag{4.21}$$

which are equivalent to condition $F_3=0$. Relations (4.21) show collinearity of $\mathbb{R}^{n-1}$–vectors $(M_{1n}, M_{2n}, \dots, M_{n-1,n})$ and $(\Gamma_1, \Gamma_2, \dots, \Gamma_{n-1})$.

4.3. Mishchenko–Fomenko flows. Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra, $\langle\cdot,\cdot\rangle$ a Ad_G-invariant scalar product on $\mathfrak{g}$. Let $a\in\mathfrak{g}$ be an arbitrary element $\mathfrak{g}_a = \{\eta\in\mathfrak{g}, [a,\eta]=0\}$ be the isotropy algebra and G_a be the adjoint isotropy group of the element a.

Let $\mathfrak{g} = \mathfrak{g}_a\oplus\mathfrak{d}$ be the orthogonal decomposition. Consider the linear operator (so called *sectional operator* [**70**]) $A_{a,b,C}:\mathfrak{g}\to\mathfrak{g}$, defined by

$$A_{a,b,C}(\xi) = \mathrm{ad}_a^{-1}\circ\mathrm{ad}_b\circ\mathrm{pr}_{\mathfrak{d}}(\xi) + C(\mathrm{pr}_{\mathfrak{g}_a}\xi), \tag{4.22}$$

where b belongs to the center of $\mathfrak{g}_a$, $\mathrm{pr}_{\mathfrak{d}}$ and $\mathrm{pr}_{\mathfrak{g}_a}$ are the orthogonal (with respect to $\langle\cdot,\cdot\rangle$) projections to $\mathfrak{d}$ and $\mathfrak{g}_a$, respectively and $C:\mathfrak{g}_a\to\mathfrak{g}_a$ is a positive definite, symmetric operator such that the quadratic form $\langle\xi, C(\xi)\rangle$ is Ad_{G_a}-invariant:

$$[\xi, C(\xi)] = 0, \quad \xi\in\mathfrak{g}_a. \tag{4.23}$$

We can always find b and C such that $A_{a,b,C}$ is positive definite. If a is regular, i.e., G_a is commutative, then the condition (4.23) is always satisfied.

Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the scalar product $\langle\cdot,\cdot\rangle$ and consider the left-trivialization: $T^*G \cong_l G \times \mathfrak{g} = \{(g,\xi)\}$. Then the quadratic form

$$H_{a,b,C} = \frac{1}{2}\langle A_{a,b,C}(\xi), \xi\rangle \tag{4.24}$$

can be regarded as the Hamiltonian of a left-invariant Riemannian metric $\kappa_{a,b,C}$ on G. After left G–reduction, the equations of the geodesic flow take the form of Euler equations

$$\dot{\xi} = [\xi, \nabla H_{a,b,C}(\xi)] = [\xi, A_{a,b,C}(\xi)], \tag{4.25}$$

that are Hamiltonian with respect to the Lie-Poisson bracket

$$\{F_1, F_2\}_1 = -\langle \xi, [\nabla F_1, \nabla F_2]\rangle. \tag{4.26}$$

The Casimir functions of the Lie-Poisson brackets are invariant polynomials.

Let $\mathcal{S}$ be the set of linear functions on $\mathfrak{g}_a$. Let $p_1, \dots, p_{\operatorname{rank} G}$ be the basic invariant polynomials of the Lie algebra $\mathfrak{g}$. Mishchenko and Fomenko proved that the polynomials obtained by shifting of argument of invariants:

$$\mathcal{C}: \qquad p_{i,\lambda}(\xi) = p_i(\xi + \lambda a), \qquad i = 1, \dots, \operatorname{rank} G, \qquad \lambda \in \mathbb{R} \tag{4.27}$$

are commuting integrals of the system (4.25) and $\{\mathcal{C}, \mathcal{S}\}_1 = 0$ [**52**]. The algebra $\mathcal{C} + \mathcal{S}$ is a complete algebra integrals of the system (4.25) [**52, 10, 70**]. We refer to systems (4.25) as *Mishchenko–Fomenko flows.*

Moreover, the system is bi-Hamiltonian (see [**51, 70**]). The second Poisson structure is linear:

$$\{F_1, F_2\}_2 = -\langle a, [\nabla F_1, \nabla F_2]\rangle, \tag{4.28}$$

with a set of Casimirs $\mathcal{S}$. The integrals $p_{i,\lambda}(\xi)$, $i = 1, \dots, \operatorname{rank} G$ are Casimir functions of the bracket $\{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_2$.

Now, let us perturb the Hamiltonian (4.24) as follows:

$$H_{a,b,C',D} = \frac{1}{2}\langle A_{a,b,C'}(\xi), \xi\rangle + \langle D(\operatorname{pr}_{\mathfrak{d}} \xi), \xi\rangle, \tag{4.29}$$

where $D : \mathfrak{d} \to \mathfrak{g}_a$ and $A_{a,b,C'}$ is given by (4.22) such that $H_{a,b,C',D}$ is positive definite. Here C' not need to satisfy (4.23).

Proposition 4.3. *The perturbed system*

$$\dot{\xi} = [\xi, \nabla H_{a,b,C',D}(\xi)] \tag{4.30}$$

has an invariant manifold

$$\operatorname{pr}_{\mathfrak{g}_a}(\xi) = 0 \tag{4.31}$$

and satisfies axioms (A1), (A2), (HP), (BP) of the systems of the Hess–Appelrot type.

5. Partial reductions

5.1. Classical Hess-Appel'rot system and spherical pendulum. Let us return to the classical problem (2.1), (2.5). Consider the motion of the rigid body in the space reference frame. Let $R \in SO(3)$ be the matrix that maps the moving reference frame to the fixed one. Following Arnol'd's notation [**6**], denote $\mathbf{m} = R\mathbf{M}, \omega = R\mathbf{\Omega}, \gamma = R\mathbf{\Gamma}$. The position of mass centre is $z_0\mathbf{e_3}$, where $\mathbf{e_3} = R\mathbf{E_3} = R(0,0,1)^T$.

Beside geometric interpretation of the conditions (2.3), Zhukovski also noticed (see [**73**]) that the motion of $\mathbf{e_3}$ is described by the spherical pendulum equation. Indeed, the Euler–Poisson equations for the variables $(\mathbf{m}, \mathbf{e_3})$, in the space frame, read

$$\dot{\mathbf{m}} = z_0\gamma \times \mathbf{e_3}, \tag{5.1}$$

$$\dot{\mathbf{e_3}} = \omega \times \mathbf{e_3} = \mathrm{pr}_{\mathbf{e_3}^\perp}\,\omega \times \mathbf{e_3}. \tag{5.2}$$

On the invariant set $M_3 = 0$ we have $\mathrm{pr}_{\mathbf{E_3}^\perp}\,\mathbf{\Omega} = (\Omega_1, \Omega_2, 0) = (J_1M_1, J_1M_2, 0) = J_1\mathbf{M}$. The relation in the space frame gives

$$\mathrm{pr}_{\mathbf{e_3}^\perp}\,\omega = J_1\mathbf{m}. \tag{5.3}$$

By differentiation of (5.2), using (5.1) and (5.3), we get

$$\ddot{\mathbf{e_3}} = J_1z_0(\gamma \times \mathbf{e_3}) \times \mathbf{e_3} + J_1^2\mathbf{m} \times (\mathbf{m} \times \mathbf{e_3}) = -J_1z_0\gamma + \mathbf{e_3}(J_1z_0\langle \mathbf{e_3}, \gamma\rangle - \langle \dot{\mathbf{e_3}}, \dot{\mathbf{e_3}}\rangle),$$

describing the Euler–Lagrange equations with multiplier $\lambda = J_1z_0\langle \mathbf{e_3}, \gamma\rangle - \langle \dot{\mathbf{e_3}}, \dot{\mathbf{e_3}}\rangle$ on the sphere $\langle \mathbf{e_3}, \mathbf{e_3}\rangle = 1$. This is the pendulum system with Lagrangian

$$l(\mathbf{e_3}, \dot{\mathbf{e_3}}) = \frac{1}{2J_1}\langle \dot{\mathbf{e_3}}, \dot{\mathbf{e_3}}\rangle - z_0\langle \gamma, \mathbf{e_3}\rangle.$$

In particular, the motion of $\mathbf{e_3}$ can be found by elliptic quadratures.

5.2. Invariant relations and reductions. Note that, considered on the whole phase space $T^*SO(3)$ of the rigid body motion, the function M_3 is the momentum mapping of the right $SO(2)$-action-rotations of the body around the line directed to the center of the mass. By the analogy with the reduction of the system to the spherical pendulum, in this subsection we study reductions of the Hamiltonian flows that satisfy axioms (A1), (A2), (HP) of the systems of the Hess–Appelrot type restricted to their invariant submanifolds. Apparently, the lowering of order in Hamiltonian systems having invariant relations was firstly studied by Levi-Civita (e.g., see [**41**, ch. X]).

Let G be a connected Lie group with a free proper Hamiltonian action on a symplectic manifold (M, ω) with the momentum map (9.9). Assume that 0 is a regular value of Φ. Let $\xi_1, \dots, \xi_p$ be the base of $\mathfrak{g}$. Then the zero level set of the momentum mapping (9.9) is given by the equations

$$M_0: \qquad \phi_i = (\Phi, \xi_i) = 0, \qquad i = 1, \dots, p. \tag{5.4}$$

Let (N_0, ω_0) be the symplectic reduced space and let $\pi_0 : M_0 \to N_0 = M_0/G$ be the canonical projection (see subsection 9.4). Consider the Hamiltonian equations:

$$\dot{x} = X_h(x). \tag{5.5}$$

Theorem 5.1 ([**36, 37**])**.** (i) *Suppose that the restriction of h to* (5.4) *is a G-invariant function. Then M_0 is an invariant manifold of the Hamiltonian system* (5.5) *and $X_h|_{M_0}$ projects to the Hamiltonian vector field X_{h_0}: $d\pi_0(X_h)|_x = X_{h_0}|_{\pi_0(x)}$, where h_0 is the induced function on N_0 defined by*

$$h|_{M_0} = \pi_0^*h_\eta = h_0 \circ \pi_0. \tag{5.6}$$

(ii) *The inverse statement also holds: if* (5.4) *is an invariant submanifold of the Hamiltonian system* (5.5)*, then the restriction of h to M_0 is a G-invariant function and $X_h|_{M_0}$ projects to the Hamiltonian vector field X_{h_0} on N_0, where h_0 is defined by* (5.6)*.*

In both cases, the Hamiltonian vector field X_h is not assumed to be G-invariant on M. Moreover $X_h|_{M_0}$ may not be G-invariant as well. It is invariant modulo the kernel of $d\pi_0$, which is sufficient the tools of symplectic reduction are still applicable.

We shall refer to the passing from $\dot{x} = X_h|_{M_0}$ to

$$\dot{y} = X_{h_0} \tag{5.7}$$

as a *partial reduction*.

The natural way to obtain a Hamiltonian h such that corresponding flow have invariant relations (5.4) is to perturb a G–invariant Hamiltonian $h_\Lambda \in C^\infty_G(M)$

$$h = h_\Lambda + \sum_{i=1}^{p} h_i \phi_i, \tag{5.8}$$

where h_i are arbitrary smooth functions on M. Then the set of $p+1$ functions $h, \phi_1, \dots, \phi_k$ satisfy axioms (A1), (A2'), (HP) of the systems of the Hess–Appelrot type, i.e., the Hamiltonian system (5.5) is *restrectively integrable* and of the form of *Hamiltonian perturbation* (see [**21**]).

The function h_Λ is G–invariant and we can perform the usual symplectic reduction to the Hamiltonian flow on the symplectic reduced space N_0. Since the Hamiltonians h and h_Λ coincide on M_0, the reduced flow is the same as the partially reduced flow (5.7). However, Hamiltonian vector fields $X_h|_{M_0}$ and $X_{h_\Lambda}|_{M_0}$ are different.

The partial reduction can be seen as a special case of the symplectic reductions studied in [**8, 43**] (see also [**42**], Theorem 14.6, Ch. III).

Theorem 5.2 ([**42**]). *Let (M, ω, h) be a Hamiltonian system and let $M_0 \subset M$ be an invariant submanifold of the vector field X_h upon which the symplectic form ω induces a 2-form ω_{M_0} of constant rank. Assume that there is a surjective submersion with connected fibres $\pi_0 : M_0 \to N_0$ onto another symplectic manifold (N_0, ω_0), which satisfies $\pi_0^*\omega_0 = \omega|_{M_0}$. Then there exist a unique reduced Hamiltonian function h_0 on N_0 such that $h|_{M_0} = h_0 \circ \pi_0$ and $d\pi_0(X_h)|_x = X_{h_0}|_{\pi_0(x)}$.*

5.3. Zhukovskii property. An immediate corollary of theorem 5.1 is

Corollary 5.1. *Suppose that the partially reduced system* (5.7) *is completely integrable in the non-commutative sense and N_0 is almost everywhere foliated on r-dimensional isotropic invariant manifolds, level sets of integrals f_i^0, $i = 1, \dots, \dim N_0 - r$. Then M_0 is almost everywhere foliated on $(r + \dim G)$-invariant isotropic manifolds*

$$\mathcal{M}_c = \{f_i = \pi_0^* f_i^0 = c_i \,|\, i = 1, \dots, \dim N_0 - r\} \tag{5.9}$$

of the system (5.5).

In the case of the classical Hess–Appel'rot system we have $M = T^*SO(3)$, $G = SO(2)$ (rotations of the body around the vector $\mathbf{E_3}$) and the reduced system is the spherical pendulum. The reduced phase phase $N_0 = T^*S^2$ is foliated on two-dimensional tori, while the invariant manifold $M_0 \subset T^*SO(3)$ defined by $M_3 = 0$ is foliated on 3-dimensional invariant Lagrangian tori.

Definition 5.1. We shall say that the Hamiltonian system (5.5) has *the Zhukovskii property* if it has invariant relations of the form (5.4) and that the reduced system (5.7) is completely integrable.

Let us make a terminological note: in [**36, 37**] for *the Zhukovskii property* has been used a different name, *partial integrability* (or *geometrical Hess–Appol'rot conditions* for natural mechanical systems). However, there is an another notion named partial integrability which has been introduced in a study based on the Poincare-Lyapounov-Nekhoroshev theorem [**60, 32, 61**] (see also [**33, 45**]). There, a Hamiltonian system

(5.5) restricted to an invariant submanifold $N \subset M$ of lower dimension is *completely integrable*, i.e., N is filled with periodic or quasi-periodic trajectories of (5.5).

The following example will illustrate the difference between the usual and partial reduction of Hamiltonian flows.

Example 5.1. Let G be a torus $\mathbb{T}^p$ and let the reduced flow be completely integrable in the commutative sense by integrals $f_1^0, \dots, f_m^0$, $m = \frac{1}{2}\dim N_0$. Consider the regular compact connected component level set of $f_1^0, \dots, f_m^0$. By Liouville's theorem, it is diffeomorphic to a m-dimensional torus $\mathbb{T}^m$ with quasi-periodic flow of (5.7). Thus the compact connected component $\hat{\mathcal{M}}_c = \pi_0^{-1}(\mathbb{T}^m)$ of (5.9) is a *torus bundle* over $\mathbb{T}^m$:

$$
\begin{tikzcd}
\mathbb{T}^p \arrow[r] & \hat{\mathcal{M}}_c \arrow[d, "\pi_0"] \\
& \mathbb{T}^m
\end{tikzcd}
\tag{5.10}
$$

Suppose that $f_1, \dots, f_m$ can be extended to commuting $\mathbb{T}^p$-invariant functions in some $\mathbb{T}^p$-invariant neighborhood V of $\hat{\mathcal{M}}_c$. Then, within V, $\hat{\mathcal{M}}_c$ is given by the equations

$$f_1 = c_1, \dots, f_m = c_m, \quad \phi_1 = 0, \dots, \phi_p = 0.$$

From the Noether theorem the functions ϕ_i commute with all $\mathbb{T}^n$-invariant functions on M and the following commuting relations hold on V:

$$\{f_a, f_b\} = \{f_a, \phi_i\} = \{\phi_i, \phi_j\} = 0,$$
$$a, b = 1, \dots, m, \quad i, j = 1, \dots, p.$$

Now, as in the case of commutative integrability of Hamiltonian systems $\hat{\mathcal{M}}_c$ is a Lagrangian torus with tangent space spanned by X_{f_a}, X_{ϕ_i}, i.e., the bundle (5.10) is trivial. However, in general, the flow of X_h over the torus $\hat{\mathcal{M}}_c$ is *not quasi-periodic*: the vector field X_h does not commute with vector fields

$$X_{f_1}, \dots, X_{f_m}, X_{\phi_1}, \dots, X_{\phi_p}$$

although Poisson brackets $\{h, f_a\}$, $\{h, \phi_i\}$ vanish on $\hat{\mathcal{M}}_c$. So, in general, we can not apply the Lie theorem [**39**] to solve the system by quadratures.

Note that, if h is G-invariant, the complete integrability of the reduced and original system are closely related [**74, 37**]. In particular, if as above G is torus, then the reconstruction problem is easily solvable by quadratures (e.g., see [**48**]). By contrary, in the case of partial reductions, although the reduced motion $y(t)$ is found by quadratures, the reconstruction problem in determining $x(t)$ $(\pi_0(x(t)) = y(t)$, $x(t_0) \in \pi_0^{-1}(y(t_0)))$ in general, leads to non-autonomous equations.

5.4. Reductions of additional symmetries. Suppose that an additional free Hamiltonian action of a connected compact Lie group K is given. Let

$$\Psi : M \to \mathfrak{k}^*$$

be the corresponding momentum mapping and let

$$\sigma : M \to M/K$$

be the natural projection. In what follows, by capital letters we shall denote the functions on $P = M/K$ and with small letters corresponding K-invariant functions on M:

$$\sigma^* : C^\infty(P) \xrightarrow{\approx} C_K^\infty(M), \qquad F \mapsto f = F \circ \sigma.$$

Since the action of K is Hamiltonian, the Poisson bracket of two K–invariant functions is K–invariant as well, so manifold $P = M/K$ carries the induced Poisson structure $\{\cdot,\cdot\}^K$ defined by $\sigma^*\{F_1, F_2\}^K = \{\sigma^* F_1, \sigma^* F_2\}$.

The symplectic leaves in $(P, \{\cdot,\cdot\}^K)$ are of the form $\Psi^{-1}(\mathcal{O}_\eta)/K$, where $\mathcal{O}_\eta$ is a coadjoint orbit of $\eta \in \mathfrak{k}^*$. The Casimir functions $I_1, \dots, I_r$ of $(P, \{\cdot,\cdot\}^K)$ are:

$$I_j = (\sigma^*)^{-1}(p_j \circ \Psi), \qquad j = 1, \dots, r = \operatorname{rank} K,$$

where $p_1, \dots, p_r$ is the base of homogeneous invariants on $\mathfrak{k}^*$, $r = \operatorname{rank} K$.

Further, suppose that the actions of G and K *commute*, that is we have $\{\phi_i, \psi_j\} = 0$, $i = 1, \dots, p$, $j = 1, \dots, q$, where

$$\psi_j = (\Psi, \zeta_j), \qquad j = 1, \dots, q.$$

and $\zeta_1, \dots, \zeta_q$ is a base of $\mathfrak{k}$. Thus ϕ_i are K-invariant and we have induced G–action on P given by the Hamiltonian functions Φ_i, $\Phi_i = (\sigma^*)^{-1}\phi_i$, $i = 1, \dots, p$.

Suppose the functions $h_\Lambda, h_1, \dots, h_p$ in the definition of the perturbed Hamiltonian (5.8) are K-invariant functions. Then we can reduce the system (5.5) to the Poisson manifold P as well:

$$\dot{F} = \{F, H\}^K, \qquad F \in C^\infty(P). \tag{5.11}$$

Here the reduced Hamiltonian H is of the form

$$H = H_\Lambda + \sum_{i=1}^{p} H_i \Phi_i, \tag{5.12}$$

where H_Λ is G–invariant, i.e., $\{H_\Lambda, \Phi_i\}^K = 0$. As a result we obtain reduced invariant relations

$$\Phi_i = 0, \quad i = 1, \dots, p \tag{5.13}$$

defining the invariant manifold $P_0 = M_0/K$ of the reduced flow (5.11).

The functions Φ_i are integrals of the non-perturbed flow so H satisfy axioms (A1), (A2') and (HP) of systems of Hess-Appel'rot type.

On the other hand, we have the induced Hamiltonian K-action on (N_0, ω_0) with the momentum mapping Ψ_0 satisfying $\Psi|_{M_0} = \Psi_0 \circ \pi_0$ (see diagram (5.14) below). Since h is K-invariant, the reduced Hamiltonian h_0 on N_0 is also K-invariant. Therefore the momentum mapping Ψ_0 is conserved along the flow of the partially reduced system (5.7).

$$\begin{tikzcd}
 & \mathfrak{k}^* & & \mathfrak{k}^* & & \mathfrak{k}^* \\
\mathfrak{g}^* & M \arrow[l, "\Phi"'] \arrow[u, "\Psi"] \arrow[d, "\sigma"'] & \supset & M_0 = \Phi^{-1}(0) \arrow[u, "\Psi"] \arrow[d, "\sigma"'] \arrow[rr, "\pi_0"] & & N_0 = M_0/G \arrow[u, "\Psi_0"] \\
 & P = M/K & \supset & P_0 = M_0/K & &
\end{tikzcd} \tag{5.14}$$

Remark 5.1. Suppose we have additional integrals of the reduced system (5.7) implying Zhukovskii property of (5.5) with respect to the G–action. Then M_0 is almost everywhere foliated on invariant isotropic manifolds $\mathcal{M}_c$ (see Corollary 5.1). Hence we obtain invariant foliation of $P_0 = M_0/K$ on manifolds of the form $\mathcal{P}_c = (K \cdot \mathcal{M}_c)/K$. It is clear that $\mathcal{P}_c$ are isotropic submanifolds of the appropriate symplectic leaves in $(P, \{\cdot,\cdot\}^K)$.

5.5. Natural mechanical systems. Let (Q, κ, v) be a natural mechanical system, where the metric κ is also regarded as a mapping $\kappa : TQ \to T^*Q$. Let G be a connected Lie group acting freely and properly on Q and $\rho : Q \to Q/G$ be the canonical projection.

For Lagrangian systems, it is convenient to work with tangent bundle reductions. Let $\mathcal{V}_q = \{\xi_q \mid \xi \in \mathfrak{g}\}$ be the tangent space to the fiber $G \cdot q$ (*vertical space at* q) and $\mathcal{V} = \cup_q \mathcal{V}_q$ be the vertical distribution. Consider the *horizontal distribution* $\mathcal{H}$ orthogonal to $\mathcal{V}$ with respect to the metric κ:

$$\mathcal{H} = \{X_q \in T_qQ \,|\, (\kappa_q(X_q), \xi_q) = 0,\ \xi \in \mathfrak{g},\ q \in Q\} \tag{5.15}$$

Let Φ be the cotangent bundle momentum mapping (9.10). Since $\mathcal{H} = \kappa^{-1}(\Phi^{-1}(0))$, the horizontal distribution is invariant with respect to the "twisted" G-action

$$g \diamond (q, X) = (g \cdot q,\ \kappa_{g\cdot q}^{-1} \circ (dg^{-1})^* \circ \kappa_q(X)), \quad X \in T_qQ, \tag{5.16}$$

that is the pull-back of canonical symplectic G-action on T^*Q via metric κ. With the above notation we get [**36**]

Theorem 5.3. (i) (Partial Noether theorem) *The horizontal distribution* (5.15) *is an invariant submanifold of the Euler–Lagrange equations* (9.4) *of the natural mechanical system* (Q, κ, v) *if and only if the potential* v *and the restriction* $\kappa_{\mathcal{H}}$ *of the metric* κ *to* $\mathcal{H}$ *are* G*-invariant with respect to the action* (5.16)*.*

(ii) (Partial Lagrange–Routh reduction) *If* $\mathcal{H}$ *is an invariant submanifold of the system* (Q, κ, v)*, then the trajectories* $q(t)$ *with velocities* $\dot q(t)$ *that belong to* $\mathcal{H}$ *project to the trajectories* $\pi(q(t)$ *of the natural mechanical system* $(Q/G, K, V)$ *with the potential* $V(\pi(q)) = v(q)$ *and the metric* K *obtained from* $\kappa_{\mathcal{H}}$ *via identification* $\mathcal{H}/G \cong T(Q/G)$*.*

Note that when κ is G-invariant, the twisted G-action (5.16) coincides with usual G-action: $g \cdot (q, X) = (g \cdot q, dg(X))$ and the induced metric K is the submersion metric.

6. Systems of Hess–Appel'rot type and Zhukovskii property

In the sequel we shall study the relationship between the Zhukovskii property and systems of Hess–Appel'rot type on the Poisson manifold $(P, \{\cdot,\cdot\}^K)$. In particular, we shall establish the dimension of invariant manifolds stated in Remark 5.1.

We say that some property holds for a *generic point* x of the manifold M if the property holds on an open dense set $U \subset M$.

6.1. Hamiltonian perturbation. Consider the system (5.11) with Hamiltonian function H of the form (5.12), where H_i are arbitrary smooth functions on P and $H_\Lambda \in C^\infty_G(P)$ is a G–invariant Hamiltonian.

Lemma 6.1. *If* F *is a* G*–invariant integral of the system*

$$\dot F = \{F, H_\Lambda\}^K, \qquad F \in C^\infty(P), \tag{6.1}$$

then it is also a integral of the system (5.11) *with perturbed Hamiltonian function* (5.12)*, restricted to the invariant manifold* (5.13)*.*

Proof. Directly, from $\{\Phi_i, F\}^K = 0$ and $\{H_\Lambda, F\}^K = 0$ we get

$$\begin{aligned}\{H, F\}^K|_{P_0} &= \{H_\Lambda, F\}^K|_{P_0} + \sum_{i=1}^p \{H_i\Phi_i, F\}^K|_{P_0} \\ &= \sum_{i=1}^p H_i\{\Phi_i, F_j\}^K|_{P_0} + \sum_{i=1}^p \Phi_i\{H_i, F\}^K|_{P_0} = 0.\end{aligned}$$

□

Therefore, if the non-perturbed flow (6.1) is integrable, for a generic perturbation (5.8), only G–invariant integrals remain to be integrals of the perturbed flow restricted to P_0.

In what follows, we suppose that at a generic point z of P_0, functions $I_1, \dots, I_r$ are independent, or equivalently, the symplectic leaf through z is regular.

Let $F_1, \dots, F_\rho$ be the Poisson–commuting integrals of the non-perturbed system (6.1), mutually independent and independent of functions $I_1, \dots, I_r$, $\Phi_1, \dots, \Phi_p$ at a generic point $z \in P_0$. Let

$$s = \dim G_z = \dim(\langle dI_1, \dots, dI_r \rangle \cap \langle d\Phi_1, \dots, d\Phi_p \rangle)|_z\,,$$

at a generic $z \in P_0$ ($G_z \subset G$ is the isotropy group of z with respect to the induced G–action) and let

$$\mathcal{F} = \{\Phi_1, \dots, \Phi_p, I_1, \dots, I_r, F_1, \dots, F_\rho\} \subset C^\infty(P), \tag{6.2}$$

$$\mathcal{F}_0 = \{\psi_1^0, \dots, \psi_q^0, f_1^0, \dots, f_\rho^0\} \subset C^\infty(N_0). \tag{6.3}$$

Here $\psi_i^0 = (\Psi_0, \zeta_i)$ are component of the momentum mapping Ψ_0 and f_i^0 are obtained from F_i by the composition

$$C_G^\infty(P) \xrightarrow{\approx} C_{G,K}^\infty(M) \xrightarrow{\imath^*} C_K^\infty(M_0) \xrightarrow{\approx} C_K^\infty(N_0), \tag{6.4}$$

where $\imath : M_0 \hookrightarrow M$ is the inclusion.

Theorem 6.1. (i)

$$2\rho \leq \dim P + 2s - 2\dim G - \operatorname{rank} K. \tag{6.5}$$

(ii) *If $\mathcal{F}$ is a complete set at a generic point $z \in P_0$ then G is commutative and the partially reduced flow* (5.7) *is completely integrable by means of integrals $\mathcal{F}_0$.*

(iii) *Contrary, suppose $\mathcal{F}_0$ is a complete set of functions on N_0. Then the manifold P_0 is almost everywhere foliated on $\frac{1}{2}(\dim P - \operatorname{rank} K)$–dimensional level sets of integrals $\mathcal{F}$, both of the non-perturbed* (6.1) *and the perturbed system* (5.11). *This is enough for integrability of the non-perturbed system* (6.1) *in the case when G is a commutative Lie group.*

Proof. I Consider Theorem 9.1 where we put (M, ω, G, Φ) to be equal to $(N_0, \omega_0, K, \Psi_0)$. We have (see [**11**]):

$$\begin{aligned} &\operatorname{dind} \Psi_0^* C^\infty(\mathfrak{k}^*) = \operatorname{dind} C_K^\infty(N_0) = \\ &\qquad = \operatorname{dind} (\Psi^* C^\infty(\mathfrak{k}^*) + C_K^\infty(N_0)) = \dim K_\mu - \dim K_y \\ &\operatorname{ddim} C_K^\infty(N_0) = \dim N_0 + \dim K_y - \dim K \end{aligned} \tag{6.6}$$

for a generic $y \in N_0$, $\mu = \Psi_0(y)$.

As above, let $p_1, \dots, p_r$ be the base of homogeneous invariants on $\mathfrak{k}^*$, $r = \operatorname{rank} K$. Then functions

$$i_i^0 = p_i \circ \Psi_0, \qquad i = 1, \dots, r$$

belong to $\Psi^* C^\infty(\mathfrak{k}^*) \cap C_K^\infty(N_0)$, and therefore belong to the centers in both algebras $\Psi^* C^\infty(\mathfrak{k}^*)$ and $C_K^\infty(N_0)$. Among them there are exactly $\operatorname{dind} C_K^\infty(N_0)$ independent ones. Under the composition (6.4), i_j^0 corresponds to the Casimir function I_j of the bracket $\{\cdot, \cdot\}^K$.

Since $F_1, \dots, F_\rho$ are mutually independent and independent of the functions $I_1, \dots, I_r$ and $\Phi_1, \dots, \Phi_p$ at a generic point $z \in P_0$, it follows that $f_1^0, \dots, f_\rho^0$ are also mutually independent and independent of the functions $i_1^0, \dots, i_r^0$ and

$$\operatorname{ddim} \{i_1^0, \cdots, i_r^0, f_1^0, \dots, f_\rho^0\} = \operatorname{dind} C_K^\infty(N_0) + \rho. \tag{6.7}$$

The set $\{i_1^0, \cdots, i_r^0, f_1^0, \dots, f_\rho^0\}$ is a commutative subset of $C^\infty_K(N_0)$, so

$$\operatorname{ddim}\{i_1^0, \cdots, i_r^0, f_1^0, \dots, f_\rho^0\} \le \frac{1}{2}(\operatorname{ddim} C^\infty_K(N_0) + \operatorname{dind} C^\infty_K(N_0)) \tag{6.8}$$

The condition that a symplectic leaf through $z \in P_0$ is regular implies that the coadjoint orbit through $\mu = \Psi_0(y)$ is regular, where $y = \pi_0(x)$, $z = \sigma(x)$, $x \in M_0$. Therefore, from (6.6) we get

$$\begin{aligned} \operatorname{ddim} C^\infty_K(N_0) + \operatorname{dind} C^\infty_K(N_0) &= \dim N_0 + \operatorname{rank} K - \dim K \\ &= \dim P - 2\dim G + \operatorname{rank} K \end{aligned} \tag{6.9}$$

Also $s = \dim G_z = \dim K_y$, for a generic $y = \pi_0(x)$, $z = \sigma(x)$, $x \in M_0$. Thus

$$\operatorname{dind} C^\infty_K(N_0) = \operatorname{rank} K - s\,. \tag{6.10}$$

Finally, combining (6.7), (6.8), (6.9) and (6.10) we obtain inequality (6.5).

II By using the proof of item (i), we get

$$\begin{aligned} &\operatorname{ddim}\mathcal{F} = \dim G + \operatorname{rank} K + \rho - s,\\ &\operatorname{dind}\mathcal{F} = \operatorname{rank} G + \operatorname{rank} K + \rho - s,\\ &\operatorname{ddim}\mathcal{F} + \operatorname{dind}\mathcal{F} = \dim G + \operatorname{rank} G + 2\operatorname{rank} K + 2\rho - 2s \end{aligned}$$

$$\le \dim G + \operatorname{rank} G + 2\operatorname{rank} K - 2s + \dim P + 2s - 2\dim G - \operatorname{rank} K \tag{6.11}$$

$$= \dim P + \operatorname{rank} G - \dim G + \operatorname{rank} K \le \dim P + \operatorname{rank} K. \tag{6.12}$$

If (6.2) is a complete set at a generic point $z \in P_0$ then we have equalities both in (6.11) and (6.12). Equality in (6.12) implies that G is a commutative group. Equality in (6.11) means that we have equality in (6.5), that is, according (6.8), $\{i_1^0, \cdots, i_r^0, f_1^0, \dots, f_\rho^0\}$ is a complete commutative subset in $C^\infty_K(N_0)$.

Then, from Theorem 9.1 we get that $\{\psi_1^0, \dots, \psi_q^0, i_1^0, \cdots, i_r^0, f_1^0, \dots, f_\rho^0\}$ is a complete set on N_0. Since i_j^0 are polynomial functions in ψ_i^0, the set (6.3) will be also a complete set of integrals of the partially reduced system (5.7).

III Suppose (6.3) is a complete set. Then $\{i_1^0, \cdots, i_r^0, f_1^0, \dots, f_\rho^0\}$ is a complete commutative subset in $C^\infty_K(N_0)$ and we have equality in (6.5). Therefore $\operatorname{ddim}\mathcal{F} = \dim G + \operatorname{rank} K + \frac{1}{2}(\dim P + 2s - 2\dim G - \operatorname{rank} K) - s = \frac{1}{2}(\dim P + \operatorname{rank} K)$. The dimension of invariant level-sets given by $\mathcal{F}$ is $\dim P - \operatorname{ddim}\mathcal{F} = \frac{1}{2}(\dim P - \operatorname{rank} K)$. $\square$

6.2. Lifting of bi–Poisson structure. As above, consider the system (5.11) with Hamiltonian function H of the form (5.12), where H_i are arbitrary smooth functions on P and $H_\Lambda \in C^\infty_G(P)$. Suppose in addition to the Poisson structure $\{\cdot,\cdot\}_1 = \{\cdot,\cdot\}^K$, that the non-perturbed system (6.1) is Hamiltonian with respect to the another Poisson structure $\{\cdot,\cdot\}_2$ which is compatible with the first one. Also, we suppose that functions Φ_i are Casimir functions of the second bracket:

$$\{\Phi_i, F\}_2 \equiv 0, \qquad i = 1, \dots, p, \quad F \in C^\infty(P). \tag{6.13}$$

Thus, the Hamiltonian flow (5.11) satisfies axioms (A1), (A2'), (HP) and (BP) of systems of Hess-Appel'rot type.

Let

$$\Pi = \{\{\cdot,\cdot\}_{\lambda_1,\lambda_2} = \lambda_1\{\cdot,\cdot\}_1 + \lambda_2\{\cdot,\cdot\}_2 \mid \lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1^2 + \lambda_2^2 \ne 0\},$$

be the corresponding pencil of compatible Poisson structures.

Theorem 6.2. *The pencil of compatible Poisson structures Π on P induces the pencil of compatible Poisson structures Π_0 within the algebra $C^\infty_K(N_0)$ of K–invariant functions on N_0*

PROOF. The Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are compatible. That is why it is enough to prove that both Poisson brackets are correctly defined within the algebra of functions $C^\infty_K(N_0)$.

The G and K actions on (M,ω) is Hamiltonian. Thus $\{\cdot,\cdot\}_1$ is well defined in all of the algebras: $C^\infty_K(M)\cong C^\infty(P)$, $C^\infty(M)$, $C^\infty_G(M)$, $C^\infty(N_0)$ and $C^\infty_K(N_0)$.

The function $F\in C^\infty(P)$ is G–invariant if and only if

$$\{F,\Phi_i\}_1=0,\qquad i=1,\dots,p. \tag{6.14}$$

The bracket $\{\cdot,\cdot\}_2$ induces the Poisson structure within $C^\infty_G(P)$ if the bracket $\{F_1,F_2\}_2$ of two G–invariant functions is again a G–invariant function.

Suppose F_1 and F_2 are G–invariant. Let us write the Jacobi identity for the bracket $\{\cdot,\cdot\}_1+\{\cdot,\cdot\}_2$ and functions F_1,F_2,Φ_i:

$$\begin{aligned}&\{\{F_1,F_2\}_1+\{F_1,F_2\}_2,\Phi_i\}_1+\{\{F_1,F_2\}_1+\{F_1,F_2\}_2,\Phi_i\}_2+\\&\{\{\Phi_i,F_1\}_1+\{\Phi_i,F_1\}_2,F_2\}_1+\{\{\Phi_i,F_1\}_1+\{\Phi_i,F_1\}_2,F_2\}_2+\\&\{\{F_2,\Phi_i\}_1+\{F_2,\Phi_i\}_2,F_1\}_1+\{\{F_2,\Phi_i\}_1+\{F_2,\Phi_i\}_2,F_1\}_2=0.\end{aligned}$$

Taking into account identities (6.13) and (6.14) for Φ_i, F_1,F_2, from the Jacobi identity we get

$$\{\{F_1,F_2\}_1,\Phi_i\}_1+\{\{F_1,F_2\}_2,\Phi_i\}_1=0. \tag{6.15}$$

Since F_1 and F_2 are G–invariant, $\{F_1,F_2\}_1$ is also G–invariant, so the first term is equal to zero. Hence, the second term in (6.15) equals zero as well and the bracket $\{\cdot,\cdot\}_2$ induces the Poisson structure within $C^\infty_G(P)\cong C^\infty_{K,G}(M)$.

The functions Φ_i are Casimir functions for the bracket $\{\cdot,\cdot\}_2$. Whence, after the restriction, we have well defined bracket $\{\cdot,\cdot\}_2$ on the submanifold (5.13). From the above considerations it follows that the Poisson bracket $\{\cdot,\cdot\}_2$ is well defined within the algebra of functions $C^\infty_G(P_0)\cong C^\infty_{G,K}(M_0)\cong C^\infty_K(N_0)$. $\square$

The pencil Π_0 can be used as a tool in proving complete integrability of the partially reduced system (5.7). For example, suppose that all structures in Π, except possible $\{\cdot,\cdot\}_2$, have the maximal rank equal to $\dim P-\operatorname{rank}K$, and that their Casimir functions are globally defined on P. Then the union $\mathcal{C}$ of Casimir functions of all Poisson structures from Π non-proportional to $\{\cdot,\cdot\}_2$ is a commutative set with respect to the all Poisson structures from Π. Moreover, let $\mathcal{S}$ be the set of Casimir functions of the bracket $\{\cdot,\cdot\}_2$. Then $\{\mathcal{C},\mathcal{S}\}_1=0$ and, since $\{\Phi_1,\dots,\Phi_p\}\subset\mathcal{S}$, the functions in $\mathcal{C}$ are G–invariant. If the pencil Π satisfies conditions of the form (9.11) (for more details see [**10**]), the set of functions $\mathcal{F}=\mathcal{C}+\mathcal{S}$ will be a complete set of functions. However, as we have seen in Theorem 6.1, even if $\mathcal{F}$ is complete, it not need to be complete at the points of the invariant set P_0.

Nevertheless, suppose that the Casimir functions $\mathcal{C}$ induce the Casimir functions $\mathcal{C}_0\subset C^\infty_K(N_0)$ of the brackets non-proportional to $\{\cdot,\cdot\}_2$. Then, from Theorems 9.3 and 9.1 we get

Proposition 6.1. *The set $\mathcal{C}_0$ is a complete commutative subset of $C^\infty_K(N_0)$ if and only if the corank of the coplexified bivectors*

$$\pi^{\mathbb{C}}_y:(\operatorname{ann}(K\cdot y))^{\mathbb{C}}\times(\operatorname{ann}(K\cdot y))^{\mathbb{C}}\to\mathbb{C}$$

is equal to $\operatorname{dind}C^\infty_K(N_0)$ *for all $\pi\in\Pi^{\mathbb{C}}_0$ at the generic point $y\in N_0$. In this case the partially reduced system* (5.7) *is integrable by means of integrals $\mathcal{F}_0=\mathcal{C}_0+\{\psi^0_1,\dots,\psi^0_q\}$.*

Here $\operatorname{ann}(K \cdot y) \subset T_y^* N_0$ is the annihilator of the tangent space to the orbit $K \cdot y$ of y.

The condition stated in Proposition 6.1 is verified in proving the complete commutative integrability of geodesic flows of normal metrics on adjoint orbits in [**12, 58**] (see Remark 7.2 given below).

7. Zhukovskii property. Examples

7.1. Magnetic flows on adjoint orbits. In order to describe partial reduction of the Hess–Appelrot rigid body system (4.1), (4.3), (4.4), (4.7) and the geodesic flows (4.30) we shall need a description of certain natural mechanical systems on adjoint orbits recently studied, in the presence of the additional magnetic force, in [**13, 14**]. Note that integrable magnetic flows on homogeneous spaces are also given in [**66, 28, 46**].

Let G be a compact connected Lie group with the Lie algebra $\mathfrak{g}$ and invariant scalar product $\langle\cdot,\cdot\rangle$. Consider the G–adjoint orbit $\mathcal{O}(a) \subset \mathfrak{g}$. The tangent space at $x = \mathrm{Ad}_g(a)$ is simply the orthogonal complement to $\mathfrak{g}_x = \{\xi \in \mathfrak{g} \,|\, [x,\xi] = 0\}$. The cotangent bundle $T^*\mathcal{O}(a)$ can be represented as a submanifold of $\mathfrak{g} \times \mathfrak{g}$:

$$T^*\mathcal{O}(a) = \{(x,p) \,|\, x = \mathrm{Ad}_g(a), p \in \mathfrak{g}_x^\perp\},$$

with the paring between $p \in T_x^*\mathcal{O}(a) \cong \mathfrak{g}_x^\perp$ and $\eta \in T_x\mathcal{O}(a)$ given by $p(\eta) = \langle p, \eta\rangle$. Then the canonical symplectic form ω on $T^*\mathcal{O}(a)$ can be seen as a restriction of the canonical linear symplectic form of the ambient space $\mathfrak{g} \times \mathfrak{g}$: $\sum_{i=1}^{\dim \mathfrak{g}} dp_i \wedge dx_i$, where p_i, x_i are coordinates of p and x with respect to some base of $\mathfrak{g}$. Let Ω be the standard Kirillov-Kostant symplectic form on $\mathcal{O}(a)$.

The canonical G–action $g \cdot (x,p) = (\mathrm{Ad}_g\, x, \mathrm{Ad}_g\, p)$ on the magnetic cotangent bundle $(T^*\mathcal{O}(a), \omega + \epsilon\rho^*\Omega)$ is Hamiltonian with the momentum mapping (see [**28, 13**])

$$m : T^*\mathcal{O}(a) \to \mathfrak{g}^* \cong \mathfrak{g}, \quad m(x,p) = [x,p] + \epsilon x. \tag{7.1}$$

A natural generalization of the magnetic spherical pendulum to the orbit $\mathcal{O}(a)$ is the mechanical system with the kinetic energy given by the *normal metric* and the potential function $V(x) = \langle c, x\rangle$, i.e., with the Hamiltonian

$$h_c(x,p) = \frac{1}{2}\langle [x,p],[x,p]\rangle + \langle c, x\rangle.$$

Another natural class of systems are the magnetic geodesic flows of the G–invariant metrics $K_{a,b}$ defined by the Hamiltonian function

$$h_{a,b}(x,p) = \frac{1}{2}\langle [b_x p],[x,p]\rangle = -\frac{1}{2}\langle \mathrm{ad}_x \mathrm{ad}_{b_x} p, p\rangle,$$

where b belongs to the center of $\mathfrak{g}_a$ and $b_x = \mathrm{Ad}_g(b)$ for $x = \mathrm{Ad}_g(a)$ [**13**]. For compact groups, we can take b such that $K_{a,b}$ is positive definite. If $b = a$ we get the Hamiltonian of the normal metric.

The equations of the magnetic pendulum, in redundant variables (x,p), are given by (see [**14**])

$$\begin{aligned} \dot{x} &= [x,[p,x]], \\ \dot{p} &= [p,[p,x]] + \epsilon[x,p] - c + \mathrm{pr}_{\mathfrak{g}_x} c, \end{aligned} \tag{7.2}$$

while the magnetic geodesic flow read (see [**13**])

$$\begin{aligned} \dot{x} &= -\mathrm{ad}_x \mathrm{ad}_{b_x} p = [[b_x,p],x], \\ \dot{p} &= -\mathrm{ad}_x^{-1}[p,[x,[b_x,p]]] + \mathrm{pr}_{\mathfrak{g}_x}[[b_x,p],p] + \epsilon[b_x,p]. \end{aligned} \tag{7.3}$$

Let $\mathcal{O}(a)$ be an arbitrary orbit.

Theorem 7.1 ([**13, 14**]). *The magnetic pendulum system, for a regular element $c \in \mathfrak{g}$, and the geodesic flows of the metrics $K_{a,b}$ on $(T^*\mathcal{O}(a), \omega + \epsilon\rho^*\Omega)$, described by equations* (7.2) *and* (7.3)*, respectively, are completely integrable by means of polynomial integrals.*

7.2. Partial reduction of rigid body systems. Consider the construction given is Section 4 for the case when the symplectic manifold M is the phase space of the n-dimensional rigid body motion about a fixed point $T^*SO(n)$. As usual, we use the left trivialization $T^*SO(n) \cong_l SO(n) \times so(n)(g, M)$, where $so(n)$ and $so(n)^*$ are identified by the use of invariant scalar product $\langle X, Y\rangle = -\frac{1}{2}\mathrm{tr}(XY)$.

We follow Ratiu's generalization of the heavy rigid body motion [**64**]. Let

$$\gamma = \gamma_{12}E_1 \wedge E_2 + \gamma_{34}E_3 \wedge E_4 + \cdots + \gamma_{k-1,k}E_{k-1} \wedge E_k, \quad k = 2[n/2] = 2\,\mathrm{rank}\,SO(n),$$

where $\gamma_{i,i+1}$ are mutually different. Also, let

$$\begin{aligned} &\chi = \chi_{12}E_1 \wedge E_2, && n > 4,\\ &\chi = \chi_{12}E_1 \wedge E_2 + \chi_{34}E_3 \wedge E_4, && n = 4, \end{aligned}$$

where $\chi_{12} \neq \chi_{34}$, $\chi_{12} \neq 0$. Then the adjoint isotropy groups of γ and χ and corresponding isotropy Lie algebras read, respectively (e.g., see [**9**])

$$\begin{aligned} &K = SO(n)_\gamma = \overbrace{SO(2) \times \cdots \times SO(2)}^{k\,times}, && \mathfrak{k} = \langle E_1 \wedge E_2, \dots, E_{k-1} \wedge E_k\rangle,\\ &G = SO(n)_\chi = SO(2) \times SO(n-2), && \mathfrak{g} = \langle E_1 \wedge E_2, E_i \wedge E_j \,|\, 3 \le i < j \le n\rangle. \end{aligned}$$

Note that K is a maximal torus in $SO(n)$, so the adjoint orbit through γ is the flag manifold $\mathcal{O}(\gamma) = SO(n)/K$, while the adjoint orbit $\mathcal{O}(\chi) = SO(n)/G = SO(n)/SO(2) \times SO(n-2)$ is Grassmannian variety $Gr^+(n, 2)$ of oriented 2-dimensional planes through the origin in $\mathbb{R}^n$.

Consider the natural left action of K and right action of G on $T^*SO(n)$. The corresponding momentum mapping Ψ and Φ, in the left trivialization are given by

$$\Psi(g, M) = \mathrm{pr}_{\mathfrak{k}}(\mathrm{Ad}_g M), \qquad \Phi(g, M) = \mathrm{pr}_{\mathfrak{g}}(M)$$

The Hamiltonian of the n–dimensional Hess–Appelrot rigid body system is the Hamiltonian perturbation of the Lagrange system (or Lagrange bi-top system for $n = 4$):

$$h(g, M) = \frac{1}{2}\langle M, \Omega\rangle + \langle \mathrm{Ad}_{g^{-1}}\gamma, \chi\rangle, \tag{7.4}$$

where $\Omega = JM + MJ$ and J is given by (4.7). The fixed element γ play the role of the horizontal vector (direction of the gravitational force) as seen in the space reference frame.

Lemma 7.1. *The Hamiltonian* (7.4) *is left K–invariant and right G–invariant on the zero level set of the momentum mapping Φ:*

$$(T^*SO(n))_0 : \ \phi_{12}(M, g) = M_{12} = 0, \ \phi_{ij}(M, g) = M_{ij} = 0, \ 3 \le i < j \le n. \tag{7.5}$$

We have $P = (T^*SO(n))/K \cong so(n) \times \mathcal{O}(\gamma)(M, \Gamma)$. The K–reduced flow is described by equations (4.1), (4.3), (4.7) (or (4.4) for $n = 4$), where one have to fix values of invariants in Γ in order that Γ belongs to the adjoint orbit $\mathcal{O}(\gamma)$

On the second hand, the symplectic reduced space $N_0 = (T^*SO(n))_0/G$ is symplectomorphic to the cotangent bundle of the adjoint orbit $\mathcal{O}(\chi) \cong Gr^+(n, 2)$.

Let $x = \operatorname{Ad}_g \chi \in \mathcal{O}(\chi)$ (in 3–dimensional case, x represents the position of the mass center in the space coordinates). We can rewrite the Hamiltonian (7.4) in the form

$$h(M, g) = \frac{J_1 + J_n}{2}\langle M, M\rangle + \phi_{12}H_{12} + \sum_{3\le i<j\le n} \phi_{ij}H_{ij} + \langle \gamma, x\rangle.$$

Therefore

$$h(M, g)|_{(T^*SO(n))_0} = \frac{J_1 + J_n}{2}\langle M, M\rangle + \langle \gamma, x\rangle.$$

After reduction to $T^*\mathcal{O}(\chi)$, the bi-invariant kinetic energy term goes to the kinetic energy of the normal metric multiplied by $(J_1 + J_n)$ (e.g, see [**9**]). Thus, likewise in the 3–dimensional case, the partially reduced flow is the pendulum system on $\mathcal{O}(\chi)$:

$$\begin{aligned} \dot x &= (J_1 + J_n)[[x, p], x], \\ \dot p &= (J_1 + J_n)[[x, p], p] - \gamma + \operatorname{pr}_{so(n)_x} \gamma, \end{aligned} \tag{7.6}$$

with Hamiltonian $h_0(x, p) = \frac{J_1+J_n}{2}\langle [x, p], [x, p]\rangle + \langle x, \gamma\rangle$ (we follow the notation of the previous section). It follows from Theorem 7.1 that the reduced system is completely integrable. Hence the system satisfies Zhukovskii property and we get the following qualitative behavior of the system,

Theorem 7.2. *The partial reduction of the Hess-Appel'rot rigid body problem defined by the Hamiltonian* (7.4) *is completely integrable pendulum type system* (7.6) *on the oriented Grassmannian variety $Gr^+(n, 2)$. The invariant manifold* (7.5) *is almost everywhere foliated by invariant* $\dim SO(n)$*-dimensional Lagrangian invariant manifolds that project to the* $2(n-2)$*-dimensional Liouville tori of the reduced system* (7.6)*.*

So, in this approach, after solving the pendulum type system, the equations of Hess-Appel'rot rigid body problem reduces to $(\dim SO(n-2) + 1)$–differential equations of the reconstruction problem.

Similarly as in subsection 5.1, it can be proved that the partial reduction of the rigid body system (4.10), (4.3), (4.15) is the pendulum system on the $(n-1)$–dimensional sphere S^{n-1} (see [**36**]), given by the Hamiltonian function

$$h_0(x, p) = \frac{J_1 + J_n}{2}(p, p) + \chi_n\langle x, \gamma\rangle.$$

Here the cotangent bundle of the sphere is realized as a submanifold $(x, x) = 1$, $(x, p) = 0$ of $\mathbb{R}^{2n}(x, p)$ and $x, \gamma \in S^{n-1}$ represent the direction of the position of the mass center χ and the position of the vertical axes Γ in the space coordinates, respectively.

7.3. Mishchenko–Fomenko flows. We follow the notation of subsections 4.3 and 5.4, where we take $(M, G, K, P, N_0) = (T^*G, G_a, G, \mathfrak{g}, T^*(G/G_a))$. Here we consider the right G_a–action and the left G–action on T^*G. The momentum maps read $\Phi(g, \xi) = \operatorname{pr}_{\mathfrak{g}_a}(\xi)$ and $\Psi(g, \xi) = \operatorname{Ad}_g \xi$, respectively.

The metric $\kappa_{a,b,c}$ is right G_a–invariant and we can project it to the homogeneous space G/G_a, that is to the adjoint orbit of a. Since we deal with the right action, the vertical distribution is left-invariant: $\mathcal{V}_g = g \cdot \mathfrak{g}_a$, while from the definition of $\kappa_{a,b,c}$, the horizontal distribution is $\mathcal{H}_g = g \cdot \mathfrak{d}$ and the submersion metric does not depend on c. The submersion metric is exactly the metric $K_{a,b}$ on $\mathcal{O}(a)$ defined above.

The Hamiltonian $H_{a,b,C',D}$ defines left–invariant metric on G that we shall denote by $\kappa_{a,b,C',D}$. The Hamiltonian functions $H_{a,b,C',D}$ and $H_{a,b,C}$ coincides on the invariant manifold

$$(T^*G)_0 \cong_l G \times \mathfrak{d}\,. \tag{7.7}$$

Therefore, the partial reduction of the geodesic flow of the metric $\kappa_{a,b,C',D}$ is the geodesic flow of the metric $K_{a,b}$. The flow (7.3) is completely integrable for any ϵ. Thus the geodesic flow of the perturbed Mishchenko–Fomenko metric $\kappa_{a,b,C,D}$ satisfies Zhukovskii property.

Remark 7.1. If a is a *regular* element of the Lie algebra $\mathfrak{g}$ (G_a is Abelian) then the invariant manifold (7.7) is almost everywhere foliated on invariant isotropic tori of the geodesic flow $\kappa_{a,b,C',D}$ and the motion over the tori is not quasi-periodic (see Example 5.1). The same holds for the left G–reduced flow (4.30) restricted to (4.31). Namely, it can be proved that $\mathcal{S} \subset \mathcal{C}$ and that $\mathcal{C}$ is complete at a generic point ξ that belongs to (4.31) (e.g, see Theorem 3.2 in [**12**]). According Lemma 6.1, the perturbed flow (4.30) has the same foliation of (4.31) on invariant tori as the non-perturbed flow (4.25). Item (ii) of Theorem 6.1 then implies the integrability of geodesic flow of the metric $K_{a,b}$.

Remark 7.2. For a *singular* $a \in \mathfrak{g}$, $\mathcal{C} + \mathcal{S}$ restricted to (4.31) is *not complete*. The complete integrability of the geodesic flow of the normal metric and the magnetic geodesic flows (7.3) on the adjoint orbit $\mathcal{O}(a)$ follows from the completeness of the commutative set $\mathcal{C}_0$ induced from (4.27) within the algebra $C^\infty_G(T^*\mathcal{O}(a))$. The completeness is obtained by verifying the condition stated in Proposition 6.1 for the pencil of compatible Poisson structures within $C^\infty_G(T^*\mathcal{O}(a))$, induced from (4.26) and (4.28), see [**12, 58, 13**]. Besides, item (iii) of Theorem 6.1 gives us an estimate of the dimension of invariant manifolds within (4.31) of the perturbed flow (4.30).

Remark 7.3. The horizontal distributions $\mathcal{H}'$ and $\mathcal{H}$ (see subsection 5.5) of the metrics $\kappa_{a,b,C,D}$ and $\kappa_{a,b,C}$ are different

$$\mathcal{H}'_g = \kappa^{-1}_{a,b,C',D}(g \cdot \mathfrak{d}) \neq g \cdot \mathfrak{d} = \mathcal{H}_g\,,$$

for $D \neq 0$, while $\mathcal{H}'_g = \mathcal{H}_g$ for $D = 0$. Let $D = 0$. In this case, the integrals $\mathcal{C}$ remains to be the integrals of the perturbed system (4.30) not only on the invariant manifold (4.31)) but on $\mathfrak{g}$ as well.

7.4. Singular Manakov flows. Similar perturbations as those of the Mishchenko–Fomenko flows can be performed for other integrable Euler equations with symmetries. The natural candidate is the singular Manakov flow. Let

$$a = (\overbrace{\alpha_1, \dots, \alpha_1}^{k_1\ times}, \dots, \overbrace{\alpha_r, \dots, \alpha_r}^{k_r\ times}), \quad b = (\overbrace{\beta_1, \dots, \beta_1}^{k_1\ times}, \dots, \overbrace{\beta_r, \dots, \beta_r}^{k_r\ times}),$$

where $k_1 + k_2 + \dots + k_r = n$, $\alpha_i \neq \alpha_j$, $\beta_i \neq \beta_j$, $i, j = 1, \dots, r$ and let

$$so(n) = \mathfrak{g} \oplus \mathfrak{d} = so(k_1) \oplus so(k_2) \oplus \dots \oplus so(k_r) \oplus \mathfrak{d} \tag{7.8}$$

be the orthogonal decomposition, where $\mathfrak{g} = \{X \in so(n) \,|\, [X, a] = 0\}$. By $M_\mathfrak{g}$ and $M_\mathfrak{d}$ we denote the projections of $M \in so(n)$ with respect to (7.8). Further, let $C : \mathfrak{g} \to \mathfrak{g}$ be an arbitrary positive definite operator. We take a and b such that the sectional operator $A_{a,b,C} : so(n) \to so(n)$ defined via

$$A_{a,b,C}(M_\mathfrak{d} + M_\mathfrak{g}) = \mathrm{ad}_a^{-1}\mathrm{ad}_b(M_\mathfrak{d}) + C(M_\mathfrak{g}), \tag{7.9}$$

is positive definite. Here ad_a and ad_b are considered as invertible linear transformations from $\mathfrak{d}$ to $[a, \mathfrak{d}] \subset Sym(n)$. Let $H_{a,b,C} = \frac{1}{2}\langle M, A_{a,b,C}(M)\rangle$. We refer to Euler equations

$$\dot{M} = [M, \Omega], \quad \Omega = \nabla H_{a,b,C}(M) = A_{a,b,C}(M), \tag{7.10}$$

as the *singular Manakov Flow.* The operator $A_{a,b,C}$ satisfies Manakov condition $[M,b]=[\Omega,a]$, so we have the Lax representation with rational parameter λ (see Manakov [**50**]):

$$\dot{L}(\lambda)=[L(\lambda),U(\lambda)],\quad L(\lambda)=M+\lambda a,\quad U(\lambda)=\Omega+\lambda b. \tag{7.11}$$

In the case the eigenvalues of a are all distinct, i.e., $\mathfrak{g}=0$, Manakov proved that the solutions of the Euler equations (7.10) are expressible in terms of θ-functions by using the algebro-geometric integration procedure developed by Dubrovin in [**24**] (see [**50**]). The explicit verification that integrals arising from the Lax representation

$$\mathcal{L}=\{\operatorname{tr}(M+\lambda a)^k \mid k=1,2,\dots,n,\ \lambda\in\mathbb{R}\}, \tag{7.12}$$

form a complete commutative set on $so(n)$ was given by Mishchenko and Fomenko [**52**].

Let us denote the set of linear functions on $\mathfrak{g}$ by $\mathcal{S}$. These are additional integrals in the case the eigenvalues of C are not all distinct and C is proportional to the identity operator, or more generally in the case C is an Ad_G-invariant, where $G=SO(k_1)\times SO(k_2)\times\cdots\times SO(k_r)\subset SO(n)$. The complete integrability of the system is proved by Bolsinov by using the pencil of compatible Poisson brackets given by the canonical Lie-Poisson bivector

$$\pi_1(\xi_1,\xi_2)|_M=-\langle M,[\xi_1,\xi_2]\rangle$$

and

$$\pi_2(\xi_1,\xi_2)|_M=-\langle M,\xi_1A\xi_2-\xi_2A\xi_1\rangle$$

(see [**10**] and [**70**], pages 241-244). The another proof is given in [**23**]. Namely, we have $\{\mathcal{L},\mathcal{S}\}_{so(n)}=0$ and $\mathcal{L}+\mathcal{S}$ is complete at a generic $M\in so(n)$.

Now, the perturbation follows the perturbation of Mishchenko–Fomenko flows:

$$\dot{M}=[M,\Omega],\quad \Omega=\nabla H_{a,b,C,D}(M), \tag{7.13}$$

where $D:\mathfrak{d}\to\mathfrak{g}$ and $H_{a,b,C,D}=H_{a,b,C}+\langle M_{\mathfrak{g}},D(M_{\mathfrak{d}})\rangle$ is positive definite (we do not suppose that C is Ad_G–invariant). The system (7.13) has the invariant manifold

$$\mathfrak{d}:\qquad M_{\mathfrak{g}}=0. \tag{7.14}$$

Besides, the restriction of the system to (7.14) has the Manakov L-A pair (7.11) and integrals (7.12).

Consider diagram (5.14), where we take

$$(M,G,K,P,N_0)=(T^*SO(n),G,SO(n),so(n),T^*(SO(n)/G)),$$

with the right G–action and the left $SO(n)$–action on $T^*SO(n)$. By the use of the map (6.4), the commutative set of function $\mathcal{L}$ induces a complete commutative set $\mathcal{L}_0$ within $C^\infty_{SO(n)}(T^*(SO(n)/G))$ [**23**]. On the other side, if G is not commutative then $\mathcal{L}+\mathcal{S}$ is not complete at $\mathfrak{d}$.

Proposition 7.1. *If G is commutative, i.e., $\alpha_i\le 2$, $i=1,\dots,r$, then the set of function $\mathcal{L}+\mathcal{S}$ is a complete set at a generic point $M\in\mathfrak{d}$.*

Proof. The proposition directly follows from item (iii) of Theorem 6.1. Alternatively, let $L_M=\{\nabla_M\operatorname{tr}(M+\lambda A)^k \mid k=1,2,\dots,n,\ \lambda\in\mathbb{R}\}$. According to (9.3), $\mathcal{L}+\mathcal{S}$ is complete at M if

$$(L_M+\mathfrak{g})^{\pi_1}\subset L_M+\mathfrak{g}. \tag{7.15}$$

The relation (7.15) is proved in [**23**] by using Theorem 9.2, namely by verifying that the dimension of the linear spaces (25) and (26) in [**23**] are equal to n. The dimension of the space (26) calculated in Lemma 2 [**23**] holds for a generic $M\in\mathfrak{d}$. On the other side, the dimension of (25) is equal to n for elements in a generic position with the property

that $M_{\mathfrak{g}}$ is a regular element in $\mathfrak{d}$ (see [**70**], pages 234-237). Since $\mathfrak{g}$ is commutative, the dimension of (25) will be n as required. □

Example 7.1. As an example, take $n = 2r$, $a = (\alpha_1, \alpha_1, \dots, \alpha_r, \alpha_r)$. Then $\mathfrak{g} = so(2) \otimes so(2) \cdots \otimes so(2)$ is the Cartan subalgebra. The set of integrals $\mathcal{L} + \mathcal{S}$ is complete at $\mathfrak{d}$ so the invariant set $\mathfrak{d}$ of the systems (7.10) and (7.13) is foliated on invariant tori. The matrix $L(\lambda)$ satisfies

$$L_{12} = L_{21} = L_{34} = L_{43} = \cdots = L_{2r-1,2r} = L_{2r,2r-1} = 0.$$

In other words, the systems (7.10), restricted to (7.14), is an example of integrable iso-holomorhic system ([**19**], see Remark 8.2 given below).

The bi-Hamiltonian formulation of singular Manakov flows can be performed by using the pencil $\Pi = \{\pi_{\lambda_1,\lambda_2} = \lambda_1\pi_1 + \lambda_2\pi_2\}$ given above (see [**10, 70**]). The singular brackets within the pencil Π are proportional to $\pi_{1,-\alpha_1}, \dots, \pi_{1,-\alpha_r}$ and the linear function on $so(k_i) \subset \mathfrak{g}$ are among the Casimirs of the brackets $\pi_{1,-\alpha_i}$, $i = 1, \dots, r$. So, if only one k_i is greater of 1 (say $k_1 = k > 1$, $k_2 = \cdots = k_r = 1$, $r = n - k + 1$, i.e., $\mathfrak{g} = so(k)$), the perturbed singular Manakov flow (7.13) satisfies axiom (BP) with respect to the second Poisson structure $\pi_{1,-\alpha_1}$. The partially reduced system is completely integrable geodesic flow on the Stiefel variety $SO(n)/SO(k)$.

8. Integration of the magnetic pendulum on $Gr^+(4,2)$

Let us consider closely the pendulum system given by the equations (7.6) in dimension four with the magnetic term added. Since the orbit $\mathcal{O}(\chi) \cong Gr^+(4,2)$ is defined with invariants, the cotangent bundle of the Grassmannian $Gr^+(4,2)$ is given by the constraints:

$$\begin{aligned}
&x_{12}^2 + x_{13}^2 + x_{14}^2 + x_{23}^2 + x_{24}^2 + x_{34}^2 = \chi_{12}^2 + \chi_{34}^2,\\
&x_{34}x_{12} + x_{23}x_{14} - x_{13}x_{24} = \chi_{12}\chi_{34},\\
&x_{12}p_{12} + x_{13}p_{13} + x_{14}p_{14} + x_{23}p_{23} + x_{24}p_{24} + x_{34}p_{34} = 0,\\
&x_{34}p_{12} + x_{23}p_{14} - x_{13}p_{24} + p_{34}x_{12} + p_{23}x_{14} - p_{13}x_{24} = 0.
\end{aligned} \tag{8.1}$$

Introducing the magnetic momentum mapping (7.1), the equations

$$\begin{aligned}
\dot{x} &= (J_1 + J_n)[x, [p, x]],\\
\dot{p} &= (J_1 + J_n)[p, [p, x]] + \epsilon[x, p] - \gamma + \mathrm{pr}_{so(4)_x}\gamma
\end{aligned}$$

become:

$$\begin{aligned}
\dot{m} &= [\gamma, x]\\
\dot{x} &= (J_1 + J_n)[m, x].
\end{aligned} \tag{8.2}$$

The equations (8.2) are special case of the equations of the completely symmetric Lagrange bitop. The Lagrange bitop is defined in [**18**] and studied in details in [**19**]. Thus, the integration procedures given in [**18, 19**] can be applied to the considered system. We will present here both of them, the classical and the algebro-geometric integration procedures.

After solving the system (8.2) one has m and x as known functions of time. In order to find p as a function of time, one needs to solve the equation

$$m = [x, p] + \epsilon x$$

in p. Let us denote

$$m_0 = m - \epsilon x = [x, p].$$

Since $\langle [p,x], so(4)_x\rangle = \langle [x, so(4)_x], p\rangle = 0$ we have $m_0 \in so(4)_x^\perp$. The operator $\mathrm{ad}_x : so(4)_x^\perp \mapsto so(4)_x^\perp$ is bijective, thus a solution $p = \mathrm{ad}_x^{-1}(m_0)$ is unique.

8.1. Classical integration of the magnetic pendulum. Starting from a well-known decomposition $so(4) = so(3) \oplus so(3)$, let us introduce as in [**19**]

$$m_1 = \frac{1}{2}(m_+ + m_-), \qquad m_2 = \frac{1}{2}(m_+ - m_-)$$

where m_+, m_- are two three-dimensional vectors which correspond to four-dimensional matrix m_{ij} according to

$$(m_+, m_-) \longmapsto \begin{pmatrix} 0 & -m_+^3 & m_+^2 & -m_-^1 \\ m_+^3 & 0 & -m_+^1 & -m_-^2 \\ -m_+^2 & m_+^1 & 0 & -m_-^3 \\ m_-^1 & m_-^2 & m_-^3 & 0 \end{pmatrix}.$$

(Similar decomposition can be performed for x, γ).

Equations (8.2) become

$$\dot m_i = 2(\gamma_i \times x_i), \qquad \dot x_i = 2(J_1 + J_n)(m_i \times x_i), \qquad i = 1,2, \tag{8.3}$$

where

$$\gamma_1 = (0,0,-\frac{1}{2}(\gamma_{12} + \gamma_{34})), \quad \gamma_2 = (0,0,-\frac{1}{2}(\gamma_{12} - \gamma_{34})).$$

If we denote $m_1 = (p_1, q_1, r_1),\ m_2 = (p_2, q_2, r_2)$, then the first group of the equations (8.3) becomes

$$\begin{aligned} \dot p_1 &= -2\gamma_{(1)3}x_{(1)2}, & \dot p_2 &= -2\gamma_{(2)3}x_{(2)2}, \\ \dot q_1 &= 2\gamma_{(1)3}x_{(1)1}, & \dot q_2 &= 2\gamma_{(2)3}x_{(2)1}, \\ \dot r_1 &= 0, & \dot r_2 &= 0, \end{aligned}$$

where we denoted with $x_{(i)j}$ the j component of the vector x_i.

The integrals of motion are for $i = 1,2$:

$$\begin{aligned} & r_i = f_{i1} \\ & (J_1 + J_n)(p_i^2 + q_i^2) + 2\gamma_{(i)3}x_{(i)3} = f_{i2} \\ & p_i x_{(i)1} + q_i x_{(i)2} + r_i x_{(i)3} = f_{i3} = \epsilon\chi_{(i)3} \\ & x_{(i)1}^2 + x_{(i)2}^2 + x_{(i)3}^2 = f_{i4} = \chi_{(i)3}^2, \end{aligned}$$

The constants f_{i3} and f_{i4} are found from the conditions (8.1).

Following [**19**] and introducing ρ_i, σ_i, defined with $p_i = \rho_i \cos\sigma_i,\ q_i = \rho_i \sin\sigma_i$, we get

$$\begin{aligned} \dot\rho_i^2 + \rho_i^2\dot\sigma_i^2 &= 4\gamma_{(i)3}^2(\chi_{(i)3}^2 - x_{(i)3}^2), \\ \rho_i^2\dot\sigma_i &= 2\gamma_{(i)3}(\epsilon\chi_{(i)3} - f_{i1}x_{(i)3}). \end{aligned}$$

It follows that $u_i = \rho_i^2$ satisfy equations

$$\dot u_i^2 = P_i(u_i),$$

where

$$P_i(u) = -(J_1 + J_n)^2 u^3 + u^2 B_i + u C_i + D_i, \qquad i = 1,2;$$

and

$$\begin{aligned} B_i &= 2f_{i2}(J_1 + J_n) - f_{i1}^2(J_1 + J_n)^2, \\ C_i &= 4\gamma_{(i)3}^2\chi_{(i)3}^2 - f_{i2}^2 + (J_1 + J_n)^2 f_{i1}(f_{i1}f_{i2} - 2\epsilon\chi_{(i)3}\gamma_{(i)3}), \\ D_i &= -(2\epsilon\gamma_{(i)3}\chi_{(i)3} - f_{i1}f_{i2})^2, \qquad i = 1,2. \end{aligned}$$

So, the integration of the system

$$\int \frac{du_1}{\sqrt{P_1(u_1)}} = t, \quad \int \frac{du_2}{\sqrt{P_2(u_2)}} = t$$

leads to the functions associated with the elliptic curves E_1, E_2 given with:

$$E_i = E_i(J_1, J_n, f_{i1}, \gamma_{(i)3}, \epsilon, \chi_{(i)3}, f_{i3}) : \qquad y^2 = P_i(u). \tag{8.4}$$

The equations (8.3) are very similar to those for the symmetric top, and they are special case of the equations of the Lagrange bitop (see [**19**]). From the equations (8.3) one concludes that the dynamics of the magnetic pendulum on $Gr^+(4,2)$ splits on two independent systems on the sphere S^2. This splitting corresponds to the fact that $Gr^+(4,2)$ is a product of two spheres. Let us mention that a general Lagrange bitop is more complex since it doesn't split on two independent Lagrange tops.

8.2. Algebro-geometric integration procedure of the magnetic pendulum. Algebro-geometric integration completely follows paper [**19**] (see also [**21**]).

The starting point in the integration is the following Lax representation:

Proposition 8.1 ([**19**])**.** *The equations* (8.2) *has the Lax representation:*

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)]$$

where $L(\lambda) = \lambda^2 c - \lambda m + x$, $A(\lambda) = \lambda\gamma - (J_1 + J_n)m$ *and* $c = \frac{1}{J_1+J_n}\gamma$

Remark 8.1. The magnetic spherical pendulum on adjoint orbits (7.2) admits a similar Lax representation that provides a complete set of commuting integrals for a regular $c \in \mathfrak{g}$ [**14**].

We will change the coordinates in order to diagonalize the matrix $\frac{1}{J_1+J_n}\gamma$. In this new basis the matrices $L(\lambda)$ have the form $\tilde{L}(\lambda) = U^{-1}L(\lambda)U$,

$$\tilde{L}(\lambda) = \begin{pmatrix} -i\Delta_{34} & 0 & -\beta_3^* - i\beta_4^* & i\beta_3 - \beta_4 \\ 0 & i\Delta_{34} & -i\beta_3^* - \beta_4^* & -\beta_3 + i\beta_4 \\ \beta_3 - i\beta_4 & -i\beta_3 + \beta_4 & -i\Delta_{12} & 0 \\ i\beta_3^* + \beta_4^* & \beta_3^* + i\beta_4^* & 0 & i\Delta_{12} \end{pmatrix}$$

where $\Delta_{12} = \lambda^2 c_{12} - \lambda m_{12} + x_{12}$, $\quad \Delta_{34} = \lambda^2 c_{34} - \lambda m_{34} + x_{34}$, and

$$\begin{aligned} \beta_3 &= x_3 + \lambda y_3, \quad x_3 = \frac{1}{2}(x_{13} + ix_{23}), \\ \beta_4 &= x_4 + \lambda y_4, \quad x_4 = \frac{1}{2}(x_{14} + ix_{24}), \\ \beta_3^* &= \bar{x}_3 + \lambda \bar{y}_3, \quad y_3 = -\frac{1}{2}(m_{13} + im_{23}), \\ \beta_4^* &= \bar{x}_4 + \lambda \bar{y}_4, \quad y_4 = -\frac{1}{2}(m_{14} + im_{24}). \end{aligned} \tag{8.5}$$

The spectral polynomial $p(\lambda, \mu) = \det\left(\tilde{L}(\lambda) - \mu \cdot 1\right)$ has the form

$$p(\lambda, \mu) = \mu^4 + P(\lambda)\mu^2 + [Q(\lambda)]^2, \tag{8.6}$$

where

$$P(\lambda) = \Delta_{12}^2 + \Delta_{34}^2 + 4\beta_3\beta_3^* + 4\beta_4\beta_4^*, \quad Q(\lambda) = \Delta_{12}\Delta_{34} + 2i(\beta_3^*\beta_4 - \beta_3\beta_4^*). \tag{8.7}$$

We can rewrite it in terms of m_{ij} and x_{ij}:

$$P(\lambda) = A\lambda^4 - B\lambda^3 + D\lambda^2 - E\lambda + F, \quad Q(\lambda) = G\lambda^4 - H\lambda^3 + I\lambda^2 - J\lambda + K. \tag{8.8}$$

Their coefficients

$$
\begin{aligned}
A &= c_{12}^2 + c_{34}^2,\\
B &= 2c_{34}m_{34} + 2c_{12}m_{12},\\
D &= m_{13}^2 + m_{14}^2 + m_{23}^2 + m_{12}^2 + m_{34}^2 + 2c_{12}x_{12} + 2c_{34}x_{34},\\
E &= 2x_{12}m_{12} + 2x_{13}m_{13} + 2x_{14}m_{14} + 2x_{23}m_{23} + 2x_{24}m_{24} + 2x_{34}m_{34},\\
F &= x_{12}^2 + x_{13}^2 + x_{14}^2 + x_{23}^2 + x_{24}^2 + x_{34}^2,\\
G &= c_{12}c_{34},\\
H &= c_{34}m_{12} + c_{12}m_{34},\\
I &= c_{34}x_{12} + x_{34}c_{12} + m_{12}m_{34} + m_{23}m_{14} - m_{13}m_{24}\\
J &= m_{34}x_{12} + m_{12}x_{34} + m_{14}x_{23} + m_{23}x_{14} - x_{13}m_{24} - x_{24}m_{13},\\
K &= x_{34}x_{12} + x_{23}x_{14} - x_{13}x_{24}
\end{aligned}
$$

are integrals of the motion. From the constraints (8.1) one can calculate values of four integrals

$$E = 2\epsilon F = 2\epsilon(\chi_{12}^2 + \chi_{34}^2), \qquad J = 2\epsilon K = 2\epsilon\chi_{12}\chi_{34}.$$

There is an involution $\sigma : (\lambda, \mu) \to (\lambda, -\mu)$ on the curve $\Gamma : p(\lambda, \mu) = 0$, which corresponds to the skew symmetry of the matrix $L(\lambda)$. Denote the factor-curve by $\Gamma_1 = \Gamma/\sigma$.

Detailed analysis of algebro-geometric properties of the curves Γ, Γ_1 one may find in [**19**].

We consider the next eigen-problem

$$\left(\frac{\partial}{\partial t} + \tilde{A}(\lambda)\right)\psi_k = 0, \quad \tilde{L}(\lambda)\psi_k = \mu_k\psi_k,$$

where ψ_k are the eigenvectors with the eigenvalue μ_k. Then $\psi_k(t, \lambda)$ form 4×4 matrix with components $\psi_k^i(t, \lambda)$. Denote by φ_i^k corresponding inverse matrix.

Let us introduce

$$g_j^i(t, (\lambda, \mu_k)) = \psi_k^i(t, \lambda) \cdot \varphi_j^k(t, \lambda)$$

(there is no summation on k) or, in other words $g(t) = \psi_k(t) \otimes \varphi(t)^k$. Matrix g is of rank 1, and we have

$$\frac{\partial \psi}{\partial t} = -\tilde{A}\psi, \qquad \frac{\partial \varphi}{\partial t} = \varphi\tilde{A}, \qquad \frac{\partial g}{\partial t} = [g, \tilde{A}].$$

We can consider vector-functions $\psi_k(t, \lambda) = \left(\psi_k^1(t, \lambda), ..., \psi_k^4(t, \lambda)\right)^T$ as one vector-function $\psi(t, (\lambda, \mu)) = \left(\psi^1(t, (\lambda, \mu)), ..., \psi^4(t, (\lambda, \mu))\right)^T$ on the curve Γ defined with $\psi^i(t, (\lambda, \mu_k)) = \psi_k^i(t, \lambda)$. The same we have for the matrix φ_i^k. The relations for the divisors of zeroes and poles of the functions ψ^i i φ_i in the affine part of the curve Γ are:

$$\left(g_j^i\right)_a = d_j(t) + d^i(t) - D_r - D_s', \tag{8.9}$$

where D_r is the ramification divisor over λ plane(see [**24**] and [**19**]) and D_s is divisor of singular points, $D_s' \le D_s$. One can easily calculate $\deg D_r = 16$, $\deg D_s = 8$.

The matrix elements $g_j^i(t, (\lambda, \mu_k))$ are meromorphic functions on the curve Γ. We need their asymptotics in the neighborhoods of the points P_k, which cover the point $\lambda = \infty$.

It was justified in [**19**] that from now on we may consider all the functions in this section as functions on the normalization $\tilde{\Gamma}$ of the curve Γ.

Let us denote by $\tilde{d}_j$ and by $\tilde{d}^i$ the following divisors:

$$\tilde{d}_1 = d_1 + P_2, \quad \tilde{d}_2 = d_2 + P_1, \quad \tilde{d}_3 = d_3 + P_4, \quad \tilde{d}_4 = d_4 + P_3,$$
$$\tilde{d}^1 = d^1 + P_2, \quad \tilde{d}^2 = d^2 + P_1, \quad \tilde{d}^3 = d^3 + P_4, \quad \tilde{d}^4 = d^4 + P_3.$$

We have (see [**19**] for details):

Proposition 8.2 ([**19**])**.** (i) *The divisors of matrix elements of g are*

$$\left(g^i_j\right) = \tilde{d}^i + \tilde{d}_j - D_r + 2\left(P_1 + P_2 + P_3 + P_4\right) - P_i - P_j$$

(ii) *The divisors $\tilde{d}_i, \tilde{d}^j$ are of the same degree*

$$\deg \tilde{d}_i = \deg \tilde{d}^j = 5.$$

Let us denote with $\Phi(t,\lambda)$ the fundamental solution of

$$\left(\frac{\partial}{\partial t} + \tilde{A}(\lambda)\right)\Phi(t,\lambda) = 0,$$

normalized with $\Phi(\tau) = 1$. Then, if we introduce functions

$$\hat{\psi}^i(t,\tau,(\lambda,\mu_k)) = \sum_s \Phi^i_s(t,\lambda) h^s(\tau,(\lambda,\mu_k))$$

where h^s are the eigenvector of $L(\lambda)$ normalized by the condition $\sum_s h^s(t,(\lambda,\mu_k)) = 1$, it follows that

$$\hat{\psi}^i(t,\tau,(\lambda,\mu_k)) = \sum_s \Phi^i_s(t,\lambda)\frac{\psi^s_k(\tau,\lambda)}{\sum_l \psi^l_k(\tau,\lambda)} = \frac{\psi^i_k(t,\lambda)}{\sum_l \psi^l_k(\tau,\lambda)}.$$

Proposition 8.3 ([**19**])**.** *The functions $\hat{\psi}^i$ satisfy the following properties*

(i) *In the affine part of $\tilde{\Gamma}$ the function $\hat{\psi}^i$ has 4 time dependent zeroes which belong to the divisor $d^i(t)$ defined by formula* (8.9)*, and 8 time independent poles, e.g.*

$$\left(\hat{\psi}^i(t,\tau,(\lambda,\mu_k))\right)_a = d^i(t) - \bar{\mathcal{D}}, \qquad \deg \bar{\mathcal{D}} = 8.$$

(ii) *At the points P_k, the functions $\hat{\psi}^i$ have essential singularities as follows:*

$$\hat{\psi}^i(t,\tau,(\lambda,\mu)) = \exp\left[-(t-\tau)R_k\right]\hat{\alpha}^i(t,\tau,(\lambda,\mu))$$

where R_k are given with

$$R_1 = i\left(\frac{\gamma_{34}}{z} - (J_1+J_n)m_{34}\right), R_2 = -R_1, R_3 = i\left(\frac{\gamma_{12}}{z} - (J_1+J_n)m_{12}\right), R_4 = -R_3$$

and $\hat{\alpha}^i$ are holomorphic in a neighborhood of P_k,

$$\hat{\alpha}^i(\tau,\tau,(\lambda,\mu)) = h^i(\tau,(\lambda,\mu)), \quad \hat{\alpha}^i(t,\tau,P_k) = \delta^k_i + v^i_k(t)z + O(z^2),$$

with

$$v^i_k = \frac{\tilde{m_{ki}}}{\tilde{c_{ii}} - \tilde{c_{kk}}}. \tag{8.10}$$

We have

Lemma 8.1 ([**19**])**.** *The following relation takes place on the Jacobian $Jac(\tilde{\Gamma})$:*

$$\mathcal{A}(d^j(t) + \sigma d^j(t)) = \mathcal{A}(d^j(\tau) + \sigma d^j(\tau))$$

where $\mathcal{A}$ is the Abel map from the curve $\tilde{\Gamma}$ to $Jac(\tilde{\Gamma})$.

From the previous Lemma we see that the vectors $\mathcal{A}(d^i(t))$ belong to some translation of the Prym variety $\Pi = Prym(\tilde{\Gamma}\,|\,\Gamma_1)$. More details concerning the Prym varieties one can find in [**57, 56, 29, 68, 69, 19**].

It was shown in [**19**] that the Baker–Akhiezer function Ψ satisfies usual conditions of normalized (n=)4-point function on the curve of genus $g = 5$ with the divisor $\bar{\mathcal{D}}$ of degree $\deg \bar{\mathcal{D}} = g + n - 1 = 8$, see [**26, 25**]. By the general theory, it should determine all dynamics uniquely.

Let us consider the differentials $\Omega^i_j = g_{ij} d\lambda, \quad i, j = 1, \dots, 4$.

It was proven by Dubrovin in the case of general position, that Ω^i_j is a meromorphic differential having poles at P_i and P_j, with residues v^i_j and $-v^j_i$ respectively. But here we have

Proposition 8.4 ([**19**]). *The four differentials Ω^1_2, Ω^2_1, Ω^3_4, Ω^4_3 are holomorphic during the whole evolution.*

Remark 8.2. The proof was based on the fact that

$$v^1_2 = v^2_1 = v^3_4 = v^4_3 = 0, \tag{8.11}$$

which is consequence of condition $\tilde{L}_{12} = \tilde{L}_{21} = \tilde{L}_{34} = \tilde{L}_{43} = 0$. It was the reason that the notion of *isoholomorphic systems* has been introduced in [**19**] to describe such class of integrable systems.

Let us recall ([**19**]) the general formulae for v:

$$v^i_j = \frac{\lambda_i \theta(A(P_i) - A(P_j) + tU + z_0)}{\lambda_j \theta(tU + z_0)\epsilon(P_i, P_j)}, \quad i \neq j, \tag{8.12}$$

where $U = \sum x^{(k)} U^{(k)}$ is certain linear combination of b periods $U^{(i)}$ of the differentials of the second kind $\Omega^{(1)}_{P_i}$, which have pole of order two at P_i; λ_i are nonzero scalars, and

$$\epsilon(P_i, P_j) = \frac{\theta[\nu](A(P_i - P_j))}{(-\partial_{U^{(i)}}\theta[\nu](0))^{1/2}(-\partial_{U^{(j)}}\theta[\nu](0))^{1/2})}.$$

(Here ν is an arbitrary odd non-degenerate characteristics.) Thus, from (8.12), it follows

Proposition 8.5 ([**19**]). *Holomorphicity of some of the differentials Ω^i_j implies that the theta divisor of the spectral curve contains some tori.*

In a case of spectral curve which is a double unramified covering

$$\pi : \tilde{\Gamma} \to \Gamma_1;$$

with $g(\Gamma_1) = g, \quad g(\tilde{\Gamma}) = 2g - 1$, as it is satisfied for the Lagrange bitop, it is really satisfied that the theta divisor contains a torus, see [**57**]. Following [**57**] and [**19**], let us denote by Π^- the set

$$\Pi^- = \left\{ L \in Pic^{2g-2}\tilde{\Gamma} \,|\, NmL = K_{\Gamma 1}, h^0(L) \text{ is odd} \right\},$$

where K_{Γ_1} is the canonical class of the curve Γ_1 and $Nm : Pic\tilde{\Gamma} \to Pic\Gamma_1$ is the norm map, see [**57, 69**] for details. For us, it is crucial that Π^- is a translate of the Prym variety Π and that Mumford's relation ([**57**], p.241-242) holds

$$\Pi^- \subset \Theta_{\tilde{\Gamma}}. \tag{8.13}$$

Let us denote

$$U = i(\chi_{34}U^{(1)} - \chi_{34}U^{(2)} + \chi_{12}U^{(3)} - \chi_{12}U^{(4)}), \tag{8.14}$$

where $U^{(i)}$ is the vector of $\tilde{b}$ periods of the differential of the second kind $\Omega^{(1)}_{P_i}$, which is normalized by the condition that $\tilde{a}$ periods are zero. We suppose here that the cycles $\tilde{a}, \tilde{b}$ on the curve $\tilde{\Gamma}$ and a, b on Γ_1 are chosen to correspond to the involution σ and the projection π, see [**69**]:

$$\pi(\tilde{a}_0) = a_0; \quad \pi(\tilde{b}_0) = 2b_0, \quad \sigma(\tilde{a}_k) = \tilde{a}_{k+2}, \quad k = 1, 2.$$

The basis of normalized holomorphic differentials $[u_0, \dots, u_5]$ on $\tilde{\Gamma}$ and $[v_0, v_1, v_2]$ on Γ_1 are chosen such that

$$\pi^*(v_0) = u_0, \quad \pi^*(v_i) = v_i + \sigma(v_i) = v_i + v_{i+2}, \quad i = 1, 2.$$

Now we have

Theorem 8.1 ([**19**]). (i) *If the vector z_0 in* (8.12) *corresponds to the translation of the Prym variety Π to Π^-, and the vector U is defined by* (8.14) *than the conditions* (8.11) *are satisfied.*

(ii) *The explicit formula for z_0 is*

$$z_0 = \frac{1}{2}(\hat{\tau}_{00}, \hat{\tau}_{01}, \hat{\tau}_{02}, \hat{\tau}_{01}, \hat{\tau}_{02}), \quad \hat{\tau}_{0i} = \int_{\tilde{b}_0} u_i, \quad i = 0, 1, 2. \tag{8.15}$$

The evolution on the Jacobian of the spectral curve, as we considered $Jac(\tilde{\Gamma})$ gives the possibility to reconstruct the evolution of the Lax matrix $L(\lambda)$ only up to the conjugation by diagonal matrices. As it was explained in [**19**], for complete integration one has to pass to the generalized Jacobian, obtained by gluing together the infinite points. Those points are P_1, P_2, P_3, P_4 and corresponding Jacobian will be denoted as $Jac(\tilde{\Gamma}| \{P_1, P_2, P_3, P_4\})$.

It can be understood as a set of classes of relative equivalence among the divisors on $\tilde{\Gamma}$ of certain degree. Two divisors of the same degree D_1 and D_2 are called *equivalent relative to the points* P_1, P_2, P_3, P_4, if there exists a function f meromorphic on $\tilde{\Gamma}$ such that $(f) = D_1 - D_2$ and $f(P_1) = f(P_2) = f(P_3) = f(P_4)$.

The generalized Abel map is defined with

$$\tilde{A}(P) = (A(P), \lambda_1(P), ..., \lambda_4(P)), \quad \lambda_i(P) = \exp \int_{P_0}^{P} \Omega_{P_iQ_0}, i = 1, ..., 4,$$

and $A(P)$ is the standard Abel map. Here $\Omega_{P_iQ_0}$ denotes the normalized differential of the third kind, with poles at P_i and at arbitrary fixed point Q_0.

We will use the generalized Abel theorem as it was formulated in [**19**]. The generalized Jacobi inverse problem can be formulated as the question of finding, for given z, points $Q_1, \dots, Q_8$ such that

$$\sum_1^8 A(Q_i) - \sum_2^4 A(P_i) = z + K,$$

$$\lambda_j = c \exp \sum_{s=1}^{8} \int_{P_0}^{Q_s} \Omega_{P_jQ_0} + \kappa_j, j = 1, ...4,$$

where K is the Riemann constant and the constants κ_j depend on the curve $\tilde{\Gamma}$, the points P_1, P_2, P_3, P_4 and the choice of local parameters around them.

We will denote by Q_s the points which belong to the divisor $\bar{\mathcal{D}}$ from the Proposition 8.3, and by E the prime form from [**29**]. Then we have

Proposition 8.6 ([**19**]). *The scalars λ_j from the formula* (8.12) *are given with*

$$\lambda_j = \lambda_j^0 \exp \sum_{k\neq j} i x^{(k)} \gamma_j^k, \qquad \lambda_j^0 = c \exp \sum_{s=1}^{8} \int_{P_0}^{Q_s} \Omega_{P_j Q_0} + \kappa_j,$$

where the vector $\vec{x} = (x^{(1)}, \dots, x^{(4)})$ denotes $t(\gamma_{34}, -\gamma_{34}, \gamma_{12}, -\gamma_{12})$ and

$$\gamma_i^j = \frac{d}{dk_j^{-1}} lnE(P_i, P)|_{P=P_j}.$$

(k_j^{-1} is a local parameter around P_j.)

To give the formulae for the Baker-Akhiezer function, we need some notations. Let

$$\alpha^j(\vec{x}) = \exp[i \sum \tilde{\gamma}_m^j x^{(m)}] \frac{\theta(z_0)}{\theta(i \sum x^{(k)} U^{(k)} + z_0)},$$

where

$$\tilde{\gamma}_m^j = \int_{P_0}^{P_j} \Omega_{P_m}^{(1)}, \quad m \neq j,$$

and $\tilde{\gamma}_m^m$ is defined by the expansion

$$\int_{P_0}^{P} \Omega_{P_m}^{(1)} = -k_m + \tilde{\gamma}_m^m + O(k_m^{-1}), \quad P \to P_m.$$

Denote

$$\phi^j(\vec{x}, P) = \alpha^j(\vec{x}) \exp(-i \int_{P_0}^{P} \sum x^{(m)} \Omega_{P_m}^{(1)}) \frac{\theta(A(P) - A(P_j) - i \sum x^{(k)} U^{(k)} - z_0)}{\theta(A(P) - A(P_j) - z_0)}.$$

Finally, one can state

Proposition 8.7 ([**19**]). *The Baker-Akhiezer function is given by*

$$\psi^j(\vec{x}, P) = \phi^j(\vec{x}, P) \frac{\lambda_j^0 \frac{\theta(A(P-P_j)-z_0)}{\epsilon(P,P_j)}}{\sum_{k=1}^4 \lambda_k^0 \frac{\theta(A(P-P_k)-z_0)}{\epsilon(P,P_k)}}, \quad j = 1, \dots, 4,$$

where z_0 is given by (8.15).

9. Appendix: Basic notions of the Hamiltonian systems

9.1. Hamiltonian systems. Let $(P, \{\cdot,\cdot\})$ be a Poisson manifold and π be the associated bivector field on P

$$\{f_1, f_2\}(x) = \pi_x(df_1(x), df_2(x)) = \sum_{i,j} \pi^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f_2}{\partial x_j}.$$

If π is non-degenerate, then the two-form $\omega = \sum \omega_{ij} dx_i \wedge dx_j$ $(\omega_{ij}\pi^{jk} = \delta_i^k)$ is a symplectic form and (P, ω) is called a symplectic manifold.

The equations:

$$\dot{x} = X_h(x) \qquad \Longleftrightarrow \qquad \dot{f} = \{f, h\}, \quad f \in C^\infty(P) \tag{9.1}$$

are called *Hamiltonian equations* with the Hamiltonian function h and $X_h^i = \sum \pi^{ij} \partial h / \partial x_j$ is the corresponding *Hamiltonian vector field*.

A function f is an *integral* of the system (constant along trajectories of (9.1)) if and only if it commutes with h: $\{h, f\} = 0$. From the Jacobi identity the Poisson bracket of two integrals is again the integral, so we can consider a Poisson subalgebra $\mathcal{F} \subset C^\infty(P)$

of integrals (or a collection of integrals closed under the Poisson bracket). Consider the linear spaces

$$F_x = \{df(x) \mid f \in \mathcal{F}\} \subset T_x^* P \tag{9.2}$$

and suppose that we can find l functionally independent functions $f_1, \dots, f_l \in \mathcal{F}$ whose differentials span F_x almost everywhere on M and that the corank of the matrix $\{f_i, f_j\}$ is equal to some constant k, i.e., $\dim\ker \pi_x|_{F_x} = k$. The numbers l and k are called *differential dimension* and *differential index* of $\mathcal{F}$ and they are denoted by $\operatorname{ddim}\mathcal{F}$ and $\operatorname{dind}\mathcal{F}$, respectively.

We say that $\mathcal{F}$ is *complete at x* if the space F_x given by (9.2) is coisotropic:

$$F_x^\pi \subset F_x\,. \tag{9.3}$$

Here F_x^π is skew-orthogonal complement of F_x with respect to π:

$$F_x^\pi = \{\xi \in T_x^* P \mid \pi_x(F_x, \xi) = 0\}.$$

The set $\mathcal{F}$ is *complete* if it is complete at a generic point $x \in P$. In this case $F_x^\pi = \ker \pi_x|_{F_x}$ and $\operatorname{dind}\mathcal{F} = \dim F_x^\pi$, for a generic $x \in P$. Equivalently, $\mathcal{F}$ is called *complete* if (see [**10, 11, 74**]):

$$\operatorname{ddim}\mathcal{F} + \operatorname{dind}\mathcal{F} = \dim P + \operatorname{corank}\{\cdot,\cdot\}.$$

The Hamiltonian system on (9.1) is *completely integrable (in noncommutative sense)* if it possesses a complete set of first integrals $\mathcal{F}$. Then (under compactness condition) P is almost everywhere foliated by $(\operatorname{dind}\mathcal{F} - \operatorname{corank}\{\cdot,\cdot\})$-dimensional invariant tori. As in the Liouville-Arnol'd theorem [**6**], the Hamiltonian flow restricted to regular invariant tori is quasi-periodic (see Nekhoroshev [**59**], Mishchenko and Fomenko [**53**] and Zung [**74**]).

9.2. Natural mechanical systems. The basic examples of Hamiltonian systems are natural mechanical systems (Q, κ, v), where Q is a configuration space, κ is a Riemannian metric on Q and $v : Q \to \mathbb{R}$ is a potential function. Let $q = (q^1, \dots, q^n)$ be local coordinates on Q. The motion of the system is described by the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial l}{\partial \dot q^i} = \frac{\partial l}{\partial q^i}, \quad i = 1, \dots, n, \tag{9.4}$$

where the Lagrangian is $l(q, \dot q) = \frac{1}{2}(\kappa_q \dot q, \dot q) - v(q) = \frac{1}{2}\sum_{ij} \kappa_{ij}\dot q^i \dot q^j - v(q)$.

Equivalently, we can pass from velocities $\dot q^i$ to the momenta p_j by using the standard Legendre transformation $p_j = \kappa_{ij}\dot q^i$. Then in the coordinates q^i, p_i of the cotangent bundle T^*Q the equations of motion read:

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i}, \qquad i = 1, \dots, n, \tag{9.5}$$

where the Hamiltonian h is the sum of the kinetic and potential energy of the system $h(q, p) = \frac{1}{2}\sum_{i,j} \kappa^{ij} p_i p_j + v(q)$. Here κ^{ij} are the coefficients of the tensor inverse to the metric.

This system of equations is Hamiltonian on T^*Q endowed with the *canonical symplectic form* $\omega = \sum_{i=1}^n dp_i \wedge dq^i$. The corresponding canonical Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}\right). \tag{9.6}$$

Let ϵ be a real parameter (a "coupling" constant). The motion of the particle under the influence of the additional magnetic field given by a closed 2-form $\epsilon\,\Omega = \sum_{1\le i<j\le n} \epsilon\, F_{ij}(q) dq^i \wedge dq^j$, is described by the following equations:

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i} + \epsilon \sum_{j=1}^{n} F_{ij} \frac{\partial H}{\partial p_j}. \tag{9.7}$$

The equations (9.7) are Hamiltonian with respect to the symplectic form $\omega + \epsilon\rho^*\Omega$, where $\rho : T^*Q \to Q$ is the natural projection. Namely, the new Poisson bracket is given by

$$\{f, g\}_\epsilon = \{f, g\} + \epsilon \sum_{i,j=1}^{n} F_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j}, \tag{9.8}$$

and the Hamiltonian equations $\dot f = \{f, h\}_\epsilon$ read (9.7).

9.3. Hamiltonian G-actions. Let a connected Lie group G act on $2n$-dimensional connected symplectic manifold (M, ω). The action is *Hamiltonian* if G acts on M by symplectomorphisms and there is a well-defined momentum mapping:

$$\Phi : M \to \mathfrak{g}^* \tag{9.9}$$

($\mathfrak{g}^*$ is a dual space of the Lie algebra $\mathfrak{g}$) such that one-parameter subgroups of symplectomorphisms are generated by the Hamiltonian vector fields of functions $\phi_\xi(y) = (\Phi(y), \xi)$, $\xi \in \mathfrak{g}$ and $\phi_{[\xi_1,\xi_2]} = \{\phi_{\xi_1}, \phi_{\xi_2}\}$. Then Φ is equivariant with respect to the given action of G on M and the co-adjoint action of G on $\mathfrak{g}^*$: $\Phi(g \cdot x) = \mathrm{Ad}^*_g(\Phi(x))$. In particular, if η belongs to $\Phi(M)$, then the co-adjoint orbit $\mathcal{O}(\eta)$ belongs to $\Phi(M)$ as well.

The mapping $f \mapsto f \circ \Phi$ is a morphism of Poisson structures: $\{f_1 \circ \Phi, f_2 \circ \Phi\}(x) = \{f_1, f_2\}_{\mathfrak{g}^*}(\eta)$, $\eta = \Phi(x)$, where $\{\cdot, \cdot\}_{\mathfrak{g}^*}$ is the Lie–Poisson bracket on $\mathfrak{g}^*$:

$$\{f_1, f_2\}_{\mathfrak{g}^*}(\eta) = (\eta, [df_1(\eta), df_2(\eta)]), \quad f_1, f_2 : \mathfrak{g}^* \to \mathbb{R}.$$

Thus, $\Phi^* C^\infty(\mathfrak{g}^*)$ is closed under the Poisson bracket. Since G acts in a Hamiltonian way, the set of G-invariant functions $C^\infty_G(M)$ in $C^\infty(M)$ is closed under the Poisson bracket as well. Also $\{\Phi^* C^\infty(\mathfrak{k}^*), C^\infty_G(M)\} = 0$ (the Noether theorem).

Suppose the group G is compact. Then we have

Theorem 9.1 ([**11**]). (i) *The algebra of functions $\Phi^* C^\infty(\mathfrak{g}^*) + C^\infty_G(M)$ is complete:*

$$\mathrm{ddim}\,(\Phi^* C^\infty(\mathfrak{g}^*) + C^\infty_G(M)) + \mathrm{dind}\,(\Psi^* C^\infty(\mathfrak{g}^*) + C^\infty_G(M)) = \dim M.$$

(ii) *Suppose $\mathcal{A} \subset C^\infty(\mathfrak{g}^*)$ is a involutive set of functions, complete on a generic coadjoint orbit $\mathcal{O}(\eta) \subset \Phi(M)$ and $\mathcal{B}$ is a complete commutative subset of $C^\infty_G(M)$:*

$$\mathrm{ddim}\,\mathcal{B} = \mathrm{dind}\,\mathcal{B} = \frac{1}{2}\left(\mathrm{ddim}\, C^\infty_G(M) + \mathrm{dind}\, C^\infty_G(M)\right).$$

Then $\Phi^*\mathcal{A} + C^\infty_G(M)$ and $\Phi^* C^\infty(\mathfrak{g}^*) + \mathcal{B}$ are complete sets on M, while $\Phi^*\mathcal{A} + \mathcal{B}$ is a complete commutative set on M.

9.4. Symplectic reductions. Let G be a Lie group with a free and proper Hamiltonian action on a symplectic manifold (M, ω) with the momentum mapping (9.9). Assume that η is a regular value of Φ, so that $M_\eta = \Phi^{-1}(\eta)$ and $M_{\mathcal{O}_\eta} = \Phi^{-1}(\mathcal{O}_\eta)$ are smooth manifolds. Here $\mathcal{O}_\eta = G/G_\eta$ is the coadjoint orbit of η. The manifolds M_η and $M_{\mathcal{O}_\eta}$ are G_η-invariant and G-invariant, respectively. There is a unique symplectic structure ω_η on $N_\eta = M_\eta/G_\eta \cong \Phi^{-1}(\mathcal{O}_\eta)/G$ satisfying

$$\omega|_{M_\eta} = d\pi^*_\eta \omega_\eta,$$

where $\pi_\eta : M_\eta \to N_\eta$ is the natural projection (Marsden and Weinstein [**47**]). According to Noether's theorem, if h is a G-invariant function, then the momentum mapping Φ is an integral of the Hamiltonian system $\dot{x} = X_h(x)$. In addition, the restriction of X_h to the invariant submanifold M_η projects to the Hamiltonian vector field X_{h_η} on the reduced space N_η with h_η defined by $h|_{M_\eta} = \pi_\eta^* h_\eta = h_\eta \circ \pi_\eta$.

9.5. Cotangent bundle reductions. As the important example, consider the G–action on the configuration space Q. The action can be naturally extended to the Hamiltonian action on (T^*Q, ω): $g \cdot (q,p) = (g \cdot q, (dg^{-1})^* p)$ with the momentum mapping Φ given by

$$(\Phi(q,p),\, \xi) = (p,\, \xi_q), \quad \xi \in \mathfrak{g}, \tag{9.10}$$

where ξ_q is the vector given by the action of one-parameter subgroup $\exp(t\xi)$ at q [**42**].

Now, let G be a connected Lie group acting freely and properly on Q and $\pi : Q \to Q/G$ be the canonical projection. Then 0 is the regular value of the cotangent bundle momentum mapping (9.10) and the reduced space $(\Phi^{-1}(0)/G, \omega_0)$ is symplectomorphic to $T^*(Q/G)$.

Suppose (Q, κ, v) is a G-invariant natural mechanical system. That is G acts by isometries and the potential is the pull back of the potential V defined on Q/G. The metric κ induce the submersion metric on Q/G (e.g., see [**7**]). The reduced system, for a zero value of the momentum mapping, is the natural mechanical system $(Q/G, K, V)$. For Abelian groups this is the classical method of Routh for eliminating cyclic coordinates [**67**]. Within Lagrangian formalism the non-Abelian construction for the zero level-set of and for the other values of the momentum mapping are given in [**5**] and [**49**], respectively.

9.6. Compatible Poisson brackets. Let $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ be compatible Poisson structures on a manifold P. In other words, each linear combination $\lambda_1\{\cdot,\cdot\}_1 + \lambda_2\{\cdot,\cdot\}_2$ with constant coefficients is again a Poisson structure (e.g., see [**10, 70, 72, 63**] and references there in). Let π_1 and π_2 be the associated bivector fields and let

$$\Pi = \{\pi_{\lambda_1,\lambda_2} \mid \lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1^2 + \lambda_2^2 \neq 0\}, \quad \pi_{\lambda_1,\lambda_2} = \lambda_1\pi_1 + \lambda_2\pi_2.$$

In what follows we shall suppose that all functions are defined on some open set U, $x \in U$. By r denote the corank of a generic bracket (or bivector) in Π at x. For each bracket in Π of corank r, we consider the set of its Casimir functions at x. Let $\mathcal{C}$ be the union of these sets. Then $\mathcal{C}$ is involutive set with respect to every Poisson bracket from Π. Let C_x denote the linear subspace of T_x^*P generated by the differentials of functions from $\mathcal{C}$. It is clear that C_x is spanned by the kernels $\ker \pi(x)$, $\pi \in \Pi$, corank $\pi(x) = r$.

Together with Π, consider its natural complexification $\Pi^{\mathbb{C}} = \{\pi_{\lambda_1,\lambda_2} = \pi_1\Lambda_1 + \pi_2\Lambda_2,$ $\lambda_1, \lambda_2 \in \mathbb{C},\ |\lambda_1|^2 + |\lambda_2|^2 \neq 0\}$. Here, we consider $\pi_{\lambda_1,\lambda_2}$ as a complex valued skew-symmetric bilinear form on the complexification of the co-tangent space $(T_x^*P)^{\mathbb{C}}$. There are only finite number of the non-proportional singular structures $\pi_{\lambda_1^1,\lambda_2^1}, \dots, \pi_{\lambda_1^\rho,\lambda_2^\rho} \in \Pi^{\mathbb{C}}$ with a corank greater then r at x. With the above notation, we can state the following remarkable result:

Theorem 9.2 (Bolsinov [**10**]). (i) *C_x^π does not depend on the choice $\pi \in \Pi$.*

(ii) $(C_x^\pi)^{\mathbb{C}} \supset C_x^{\mathbb{C}} + \ker \pi_{\lambda_1^1,\lambda_2^1}(x) + \dots + \ker \pi_{\lambda_1^\rho,\lambda_2^\rho(x)}$.

(iii) $(C_x^\pi)^{\mathbb{C}} = C_x^{\mathbb{C}} + \ker \pi_{\lambda_1^1,\lambda_2^1}(x) + \dots + \ker \pi_{\lambda_1^\rho,\lambda_2^\rho(x)}$ *if and only if*

$$\dim_{\mathbb{C}} K_{\lambda_1^i,\lambda_2^i} = r, \quad K_{\lambda_1^i,\lambda_2^i} = \ker \pi_0|_{\ker \pi_{\lambda_1^i,\lambda_2^i}} \subset (T_x^*C)^{\mathbb{C}}, \quad i = 1,\dots,\rho, \tag{9.11}$$

where $\pi_0 \in \Pi$ is of the maximal rank at x.

As a corollary, an important completeness condition is formulated in [**10**]:

Theorem 9.3 ([**10**]). *Let* $\pi \in \Pi$ *and* corank $\pi(x) = r$. *Then* $\mathcal{C}$ *is a complete commutative set at* $x \in P$ *if and only if* corank $\pi'(x) = r$ *for all* $\pi' \in \Pi^{\mathbb{C}}$, $\pi' \neq \lambda\pi$, $\lambda \in \mathbb{C}$.

By the use of Theorem 9.2 one can also formulate conditions for non-commutative integrability in the case that some of the brackets in Π are not of the maximal rank [**10**] (see also [**63**]).

Acknowledgments. The research was supported by the Serbian Ministry of Science Project 144014 Geometry and Topology of Manifolds and Integrable Dynamical Systems.

References

[1] M. Adler and P. van Moerbeke, *Linearization of Hamiltonian Systems, Jacobi Varieties and Representation Theory*, Advances in Math.**3**8, 318-379 (1980).

[2] M. Adler and P. van Moerbeke, *The complex geometry of the Kowalewski -Painlevé analysis*, Invent. Math. **97**, 3-51 (1989).

[3] M. Adler, P. van Moerbeke and P. Vanhaeke, *Algebraic integrability, Painlevé geometry and Lie algebras*, Springer-Verlag, Berlin, 2004.

[4] G. G. Appel'rot, *The problem of motion of a rigid body about a fixed point*, Uchenye Zap. Mosk. Univ. Otdel. Fiz. Mat. Nauk **11** (1894) 1–112 (in Russian).

[5] V. I. Arnol'd, V. V. Kozlov and A. I. Neishtadt, *Mathematical aspects of classical and celestial mechanics*, Itogi Nauki i Tekhniki. *Sovr. Probl. Mat. Fundamental'nye Napravleniya*, Vol. 3, VINITI, Moscow 1985. English transl.: Encyclopadia of Math. Sciences, Vol.3, Springer-Verlag, Berlin 1989.

[6] V. I. Arnol'd, *Mathematical methods of classical mechanics*, Springer-Verlag, 1978.

[7] A. V. Beljaev, *Motion of a multidimensional rigid body with a fixed point in a gravitational force field*, Mat. Sb. **114(156)**(1981) no. 3, 465-470 (Russian).

[8] S. Benenti and W. M. Tulczyjew, *Remarques sur les réductions symplectiques*, C. R. Acad. Sci. Paris, **294** Série I (1982) 561–564.

[9] A. Besse, *Einstein Manifolds*, Springer, A Series of Modern Surveys in Mathematics, 1987.

[10] A. V. Bolsinov, *Compatible Poisson brackets on Lie algebras and the completeness of families of functions in involution*, Izv. Acad. Nauk SSSR, Ser. matem. **55** (1991), no.1, 68–92 (Russian); English translation: Math. USSR-Izv. **38** (1992), no.1, 69–90.

[11] A. V. Bolsinov and B. Jovanović, *Non-commutative integrability, moment map and geodesic flows*, Annals of Global Analysis and Geometry **23**, no. 4, 305–322 (2003), arXiv: math-ph/0109031.

[12] A. V. Bolsinov and B. Jovanović, *Complete involutive algebras of functions on cotangent bundles of homogeneous spaces*, Mathematische Zeitschrift **246** (2004), no. 1–2, 213–236.

[13] A. V. Bolsinov and B. Jovanović, *Magnetic Geodesic Flows on Coadjoint Orbits*, J. Phys. A: Math. Gen. **39** (2006), L247–L252, arXiv: math-ph/0602016.

[14] A. V. Bolsinov and B. Jovanović, *Magnetic Flows on Homogeneous Spaces*, Com. Mat. Helv., **83** (2008), no. 3, 679700, arXiv: math-ph/0609005.

[15] A. V. Borisov and I. S. Mamaev, *The Hess case in the dynamics of a rigid body*, Prikl. Mat. Mekh. **67** (2003) no. 2, 256–265 (Russian); English transl. J. Appl. Math. Mech. **67** (2003) no. 2, 227–235.

[16] S. G. Dalalyan, *Prym varieties of unramified double coverings of the hyperelliptic curves*, Uspekhi Math. Naukh **29**, 165-166 (1974), [in Russian].

[17] V. Dragović, *Algebro-geometric integration in classical and statistical mechanics*, Selected Topics. Stanković, Bogoljub (ed.), Three topics from contemporary mathematics. Beograd: Matematički Institut SANU. Zb. Rad., Beogr. **11 (19)** (2006) 121-154.

[18] V. Dragović and B. Gajić, *An L-A pair for the Hess–Appel'rot system and a new integrable case for the Euler-Poisson equations on* $\mathrm{so}(4) \times \mathrm{so}(4)$, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), 845–855; arXiv math-ph/9911047.

[19] V. Dragović and B. Gajić, *The Lagrange bitop on* $\mathrm{so}(4) \times \mathrm{so}(4)$ *and geometry of the Prym varieties*, Amer. J. Math. **126** (2004) no. 5, 981-1004, arXiv: math-ph/0201036.

[20] V. Dragović and B. Gajić, *Matrix Lax polynomials, geometry of Prym varieties and systems of Hess-Appelrot type*, Lett. Math. Phys. **76** (2006), no. 2-3, 163–186.

[21] V. Dragović and B. Gajić, *Systems of Hess–Appel'rot type*, Comm. Math. Phys. **256** (2006), 397–435; arXiv: math-ph/0602022.

[22] V. Dragović, B. Gajić *Elliptic curves and a new construction of integrable systems* arXiv:0901.4743, Regular and Chaotic Dynamics, **14**, No 4-5, 2009, p. 360-372.

[23] V. Dragović, B. Gajić and B. Jovanović *Singular Manakov Flows and Geodesic Flows of Homogeneous Spaces of SO(n)*, Transfomation Groups (2009), arXiv: 0901.2444v1 [math-ph].

[24] B. A. Dubrovin, *Completely integrable Hamiltonian systems connected with matrix operators and Abelian varieties* Func. Anal. and its Appl, **1**1 (1977), 28-41, [in Russian].

[25] B. A. Dubrovin, *Theta-functions and nonlinear equations* Uspekhi Math. Nauk **3**6, 11-80 (1981) [in Russian].

[26] B. A. Dubrovin, I. M. Krichever and S. P. Novikov, *Integrable systems I.* In: *Dynamical systems IV.* Berlin: Springer-Verlag, 1990, pp.173-280

[27] B. A. Dubrovin, V. B. Matveev and S. P. Novikov, *Nonlinear equations of Kortever-de Fries type, finite zone linear operators and Abelian varieties* Uspekhi Math. Nauk **3**1, 55-136 (1976) [in Russian].

[28] D. I. Efimov, *The magnetic geodesic flows on a homogeneous symplectic manifold*, Siberian Math. J. **46**(1) (2005), 83–93.

[29] J. D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, **3**52, Springer-Verlag, 1973.

[30] B. Gajić, *Dynamics and geometry of integrable rigid body motion*, Proceedings of 2nd International Congress of Serbian Society of Mechanics, Palic 1-5 June 2009, 103-127.

[31] L. Gavrilov and A. Zhivkov, *The complex geometry of Lagrange top* L'Enseignement Mathématique **4**4, 133-170 (1998).

[32] G. Gaeta, *The Poincare-Lyapounov-Nekhoroshev theorem*, Ann. Phys. **297** (2002), 157–173.

[33] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *Bi-Hamiltonian partially integrable systems*, J. Math. Phys. **44**(5) (2003), 1984–1997, arXiv: math.DS/0211463.

[34] V. V. Golubev, *Lectures on integration of the equations of motion of a rigid body about a fixed point*, Moskva, Gostenhizdat, 1953 (in Russian); English translation: Transl. Philadelphia, PA: Coronet Books, 1953.

[35] W. Hess, *Über die Euler'schen Bewegungsgleichungen und über eine neue particuläre Lösung des Problems der Bewegung eines starren Körpers un einen festen Punkt*, Math. Ann. **37**(2) (1890), 178–180.

[36] B. Jovanović, *Partial Reduction of Hamiltonian Flows and Hess–Appelrot Systems on* $\mathrm{SO}(n)$, Nonlinearity **20** (2007), 221–240; arXiv:math-ph/0611062.

[37] B. Jovanović, *Symmetries and Integrability*, Publ. Inst. Math. (Beograd) (N.S.) **84(98)** (2008), 1–36.

[38] S. Kowalevski, *Sur le problème de la rotation d'un corps solide autour d'un point fixe* Acta Math. **1**2, 177–232 (1889).

[39] V. V. Kozlov, *Symmetries, topology, resonansies in Hamiltonian mechanics.* Izevsk, 1995, p. 429 [in Russian].

[40] E. Leimanis, *The general problem of the motion of coupled rigid bodies about a fixed point.* Berlin, Heidelberg, New York: Springer-Verlag, 1965.

[41] T. Levi-Civita and V. Amaldi, *Lezioni di Meccanica Razionale*, Volume Secondo, Parte Seconda, Bologna, 1927.

[42] P. Libermann and C.M. Marle, *Symplectic geometry and analytic mechanics*, Math. Appl. **35**, Reidel, Boston, 1987.

[43] A. Lichnerowicz, *Variété symplectique et dynamique associée à une sous-variété*, C. R. Acad. Sci. Paris, Sér. A **280** (1975) 523–527.

[44] J. Liouville, *Note sur lintégration des équations differentielles de la dynamique*, présentée au bureau des longitudes le 29 juin 1853, J. Math Pures Appl. 20 (1855), 137–138.

[45] A. J. Maciejewski, M. Przybylska and H. Yoshida, *Necessery conditions for partial and super-integrability of Hamiltonian systems with homogeneous potential*; arXiv: nlin.SI/0701057.

[46] A. A. Magazev, I. V. Shikorov and Yu. A. Yurevich, *Integrable Magnetic Geodesic Flows on Lie Groups*, Teor. Matem. Fiz. **156**, No. 2, (2008) 189–206 (Russian); English translation: Theoretical and Mathematical Phusics **156**, No. 2, (2008) 1127–1141.

[47] J. E. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5** (1974) 121–130.

[48] J. E. Marsden, R. Montgomery and T. Ratiu, *Reduction, symmetry and phases in mechanics*, Memoirs of the American Mathematical Society, volume 88, number 436, Providence, 1990.

[49] J. E. Marseden, T. S. Ratiu and J. Scheurle, *Reduction theory and the Lagrange–Routh reduction*, J. Math. Phys. **41**(6) (2000), 3379–3429.

[50] S. V. Manakov, *Note on the integrability of the Euler equations of* n*-dimensional rigid body dynamics*, Funkc. Anal. Pril. **10**(4) (1976), 93–94 (Russian).

[51] M. V. Mescherkov, *On the property of the multidimensional rigid body inertial tensor*, Usp. Mat. Nauk **38** (1983) no. 5, 201–202 (Russian).

[52] A. S. Mishchenko and A. T. Fomenko, *Euler equations on finite-dimensional Lie groups*, Izv. Akad. Nauk SSSR, Ser. Mat. **42**(2) (1978), 396–415 (Russian); English translation: Math. USSR-Izv. **12**(2) (1978), 371–389.

[53] A. S. Mishchenko and A. T. Fomenko, *Generalized Liouville method of integration of Hamiltonian systems*, Funkts. Anal. Prilozh. **12**(2) (1978), 46–56 (Russian); English translation: Funct. Anal. Appl. **12** (1978), 113–121.

[54] A. S. Mishchenko and A. T. Fomenko, *Integration of Hamiltonian systems with noncommutative symmetries*, Tr. Semin. Vekt. Tenz. Anal. Prilozh. Geom. Mekh. Fiz. **20** (1981), 5–54 (Russian).

[55] P. van Moerbeke and D. Mumford, *The spectrum of difference operators and algebraic curves* Acta Math. **1**43, 93-154 (1979).

[56] D. Mumford, *Theta characteristics of an algebraic curve* Ann. scient. Ec. Norm. Sup. 4 serie **4**, 181-192 (1971).

[57] D. Mumford, *Prym varieties 1* A collection of papers dedicated to Lipman Bers, New York: Acad. Press 325-350 (1974).

[58] I. V. Mykytyuk and A. Panasyuk, *Bi-Poisson structures and integrability of geodesic flows on homogeneous spaces*, Transformation Groups **9**(3) (2004), 289–308.

[59] N. N. Nekhoroshev, *Action-angle variables and their generalization*, Tr. Mosk. Mat. O.-va. **26** (1972), 181–198, (Russian); English translation: Trans. Mosc. Math. Soc. **26** (1972), 180–198.

[60] N. N. Nekhoroshev, *The Poincare–Lyapounov–Liouville–Arnold theorem*, Funct. Anal. Appl. 28 (1994), 128–129.

[61] N. N. Nekhoroshev, *Types of integrability on a submanifold and generalizations of Gordon's theorem*, Tr. Mosk. Mat. Obs. **66** (2005), 184–262 (Russian); English translation: Trans. Moscow Math. Soc. (2005), 169–241.

[62] P. A. Nekrasov, *Analytic investigation of a certain case of motion of a heavy rigid body about a fixed point* Mat. Sbornik **1**8, 161-274 (1895).

[63] A. Panasyuk, *Bi-Hamiltonian structures with symmetries, Lie pencils and integrable systems*, J. Phys. A: Math. Theor. **42** (2009) 16205 (20 pp).

[64] T. S. Ratiu, *Euler-Poisson equations on Lie algebras and the N–dimensional heavy rigid body*, Amer. J. Math. **104** (1982) 409-448.

[65] T. Ratiu, T and P. van Moerbeke, *The Lagrange rigid body motion* Ann. Ins. Fourier, Grenoble **3**2, 211-234 (1982).

[66] A. G. Reyman and M. A. Semenov-Tian-Shanski, *Group theoretical methods in the theory of finite dimensional integrable systems*, In: V. I. Arnol'd, S. P. Novikov (eds.), *Dynamical Systems VII*, Springer-Verlag 1994, pp. 116–225

[67] E. J. Routh, *Treatise on the Dynamics of a System of Rigid Bodies*, MacMillan, London 1860.

[68] V. V. Shokurov, *Algebraic curves and their Jacobians*, In: Algebraic Geometry III, Berlin: Springer-Verlag, 1998, pp.219-261

[69] V. V. Shokurov, *Distinguishing Prymians from Jacobians*, Invent. Math. **6**5, 209-219 (1981).

[70] V. V. Trofimov and A. T. Fomenko, *Algebra and Geometry of Integrable Hamiltonian Differential Equations*, Moskva, Faktorial, 1995 (Russian).

[71] H. Yoshida, *Necessary conditions for the existence of algebraic first integrals, I: Kowalevski's exponents* J. Celest. Mech. **3**1, 363-379 (1983).

[72] I. Zakharevich, *Kronecker Webs, Bihamiltonian Structures and the Method of Argument Translation*, Transform. Groups **6** (2001) 267–300, arXiv: math.SG/9908034.

[73] N. E. Zhukovski, *Geometrische Interpretation des Hess'schen Falles der Bewegung eines schweren starren Körpers um einen festen Punkt*, Jber. Deutchen Math. Verein. **3** (1894) 62–70.

[74] N. T. Zung, *Torus actions and integrable systems*, In: A. V. Bolsinov, A. T. Fomenko, A. A. Oshemkov (eds.), *Topological Methods in the Theory of Integrable Systems*, Cambridge Scientific Publ., 2006; arXive: math.DS/0407455.

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